Two-point Padé approximant to power law cdf

Here is the cdf. The closed-form expression involves harmonic numbers of order p, $H_n^{(p)}$.

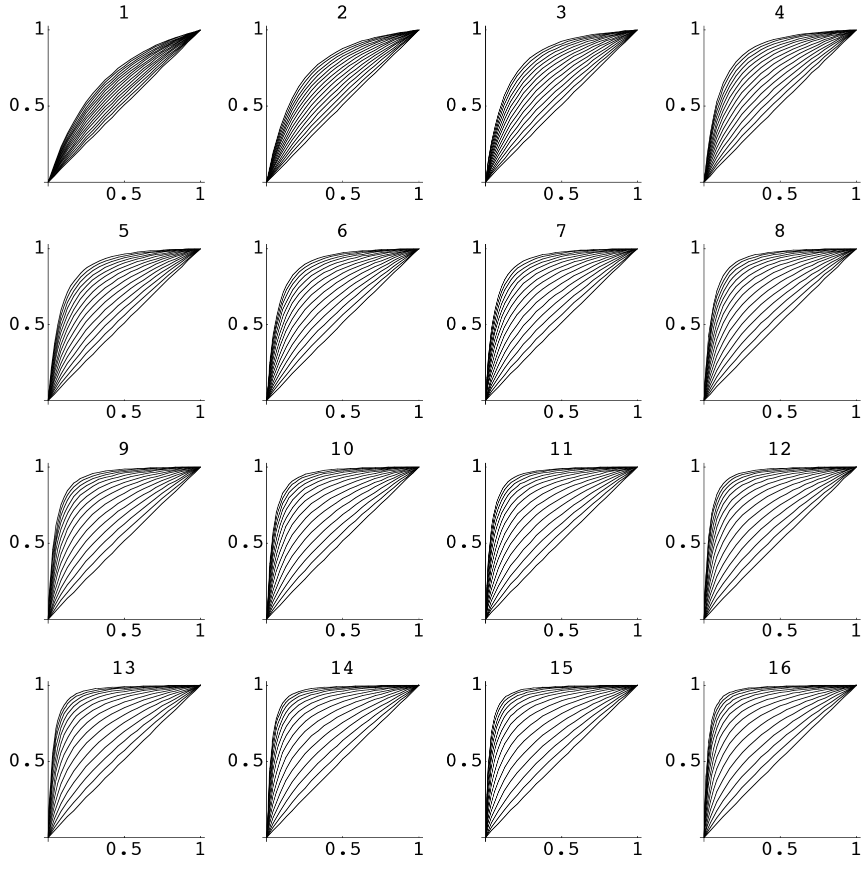
$$\textbf{cdf[p\_, n\_, x\_]} = \frac{\sum_{k=1}^{x} k^{-p}}{\sum_{k=1}^{n} k^{-p}}$$

$$\frac{H_x^{(p)}}{H_n^{(p)}}$$

Visualize the cdf for $1 \leq n \leq 12$.

<< Graphics`

DisplayTogetherArray[Partition[Table[Plot[Evaluate[Table[cdf[*p*, *n*, *n x*], {*p*, 0.05, 3, 0.2}]], {*x*, 0, 1}, PlotLabel → *n*, AspectRatio → Automatic, Ticks → {{0, 0.5, 1}, {0, 0.5, 1}}], {*n*, 1, 16}], 4]]



Determine a two-point Padé approximant to the *inverse* cdf of the form $y\,(a + b\,y)/(1 + c\,y + d\,y^2)$.

$$\textbf{InverseSeries[cdf[}p, n, n\,x\textbf{]} + O[x]^3, y\textbf{]} - \frac{y\,(a + b\,y)}{1 + c\,y + d\,y^2} \textbf{ // Simplify}$$

$$\left(\frac{H_n^{(p)}}{n\,p\,\zeta(p+1)} - a\right) y + \left(\frac{(p+1)\,\zeta(p+2)\,(H_n^{(p)})^2}{2\,n\,p^2\,\zeta(p+1)^3} - b + a\,c\right) y^2 + O(y^3)$$

InverseSeries[cdf[*p*, *n*, *n x*] + *O*[*x*, 1]^2, *y*] − $\frac{y(a+b\,y)}{1+c\,y+d\,y^2}$ // Simplify

$$\left(1-\frac{a+b}{c+d+1}\right)+\left(\frac{a(d-1)-b(c+2)}{(c+d+1)^2}-\frac{H_n^{(p)}}{n\,p\,H_n^{(p+1)}-n\,p\,\zeta(p+1)}\right)(y-1)+O((y-1)^2)$$

cdf /: cdf$^{(-1)}$ = Function[{*p*, *n*, *y*}, Evaluate[$\frac{y(a+b\,y)}{1+c\,y+d\,y^2}$ /. First@Solve[{% == 0, %% == 0}, {*a*, *b*, *c*, *d*}]]];

Here are the inverse cdf approximant for $p = 0.7$, $n = 10$,

cdf$^{(-1)}$[0.7, 10, *y*]

$$\frac{(0.192932\,y+0.276153)\,y}{0.226463\,y^2-0.757378\,y+1}$$

$p = 0.7$, $n = 20$,

cdf$^{(-1)}$[1.5, 20, *y*]

$$\frac{(0.0539372-0.0318984\,y)\,y}{0.746003\,y^2-1.72396\,y+1}$$

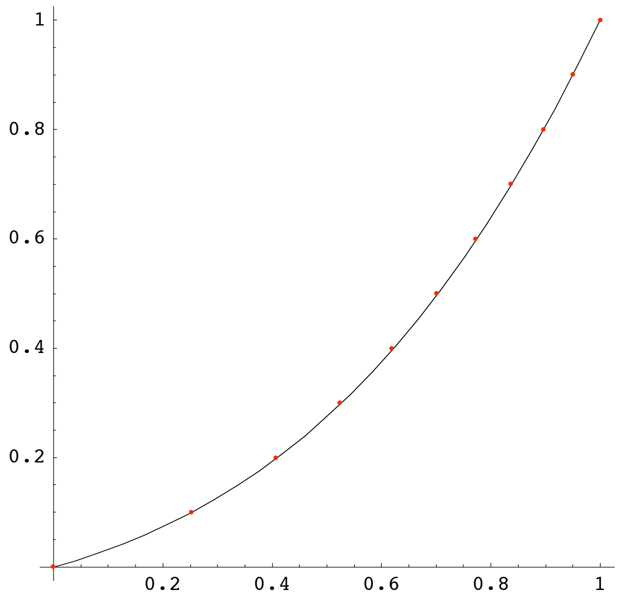
and $p = 3$, $n = 5$.

cdf$^{(-1)}$[3., 5, *y*]

$$\frac{(0.0730319-0.0644243\,y)\,y}{0.590435\,y^2-1.58183\,y+1}$$

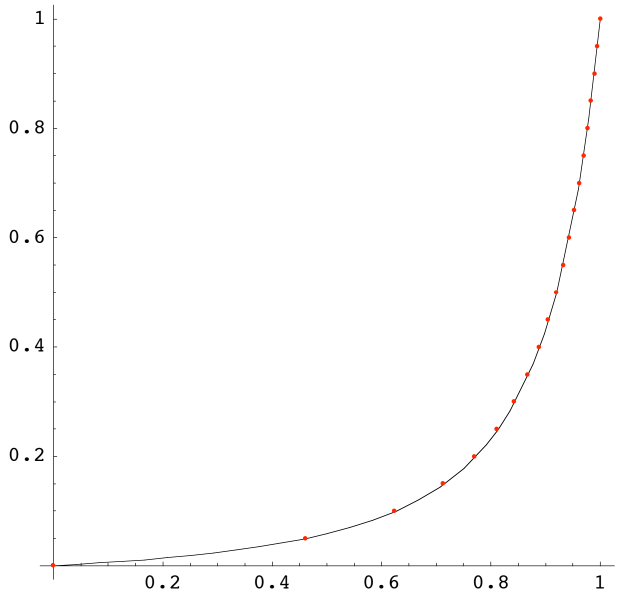
Here is a plot of these approximants, compared to the sample (inverse) cdf.

**With[{*n* = 10, *p* = 0.7}, Plot[cdf$^{(-1)}$[*p*, *n*, *y*], {*y*, 0, 1},
AspectRatio → Automatic, Epilog → {Red, Point /@ Table[{cdf[*p*, *n*, *n x*], *x*}, {*x*, 0, 1, 0.1}]}]]**



- Graphics -

**With[{n = 20, p = 3 / 2}, Plot[cdf$^{(-1)}$[p, n, y], {y, 0, 1},
AspectRatio → Automatic, Epilog → {Red, Point /@ Table[{cdf[p, n, $n\,x$], x}, {x, 0, 1, 0.05}]}]]**



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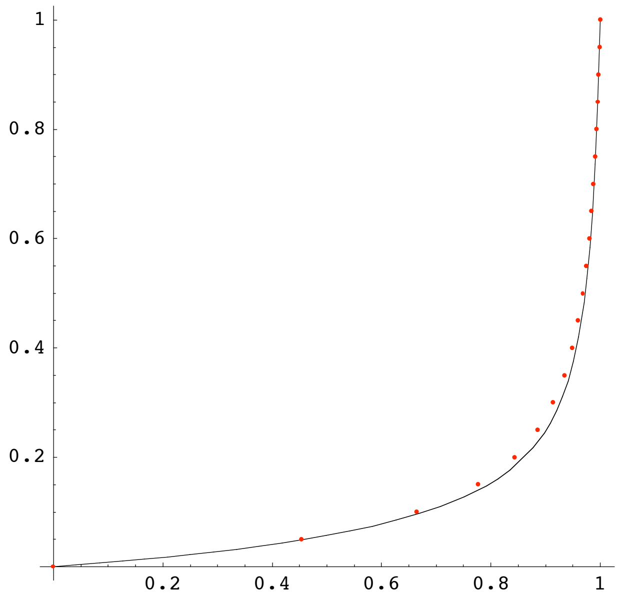
**With[{n = 5, p = 3}, Plot[cdf$^{(-1)}$[p, n, y], {y, 0, 1}, AspectRatio → Automatic,
Epilog → {Red, Point /@ Table[{cdf[p, n, $n\,x$], x}, {x, 0, 1, 0.05}]}]]**



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The agreement is very good.

As a check, compute cdf$^{(-1)}$(cdf) and cdf(cdf$^{(-1)}$) for $n = 20$ and $p = 0.5$.

With[{n = 20, p = 0.5}, Table[{cdf$^{(-1)}$[p, n, cdf[p, n, n x]], x}, {x, 0, 1, 0.05}]]

0	0
0.0497781	0.05
0.0992597	0.1
0.148727	0.15
0.198284	0.2
0.247964	0.25
0.297767	0.3
0.347684	0.35
0.397701	0.4
0.447801	0.45
0.497968	0.5
0.548186	0.55
0.598437	0.6
0.648707	0.65
0.698982	0.7
0.749247	0.75
0.799489	0.8
0.849697	0.85
0.899859	0.9
0.949963	0.95
1.	1.

With[{n = 20, p = 3 / 2}, Table[{cdf[p, n, n cdf$^{(-1)}$[p, n, x]], x}, {x, 0, 1, 0.05}]]

0	0
0.049997	0.05
0.0999764	0.1
0.149922	0.15
0.199822	0.2
0.249664	0.25
0.299444	0.3
0.34916	0.35
0.398817	0.4
0.448426	0.45
0.498007	0.5
0.547587	0.55
0.597203	0.6
0.646901	0.65
0.696736	0.7
0.746765	0.75
0.797046	0.8
0.847617	0.85
0.898464	0.9
0.949432	0.95
1.	1.