$\int_0^\infty K_0\left(\sqrt{x^2+y^2}\right) e^{-\alpha x} \, dx$

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■ Definition

Define

$$k(\lambda, \alpha, y) = \int_0^\infty K_0\left(\lambda \sqrt{x^2+y^2}\right) e^{-\alpha x} \, dx.$$

■ Direct Integration

Mathematica does not compute the integral $G(\alpha, \beta) = k(1, \alpha, \beta)$ directly.

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) e^{-\alpha x} \, dx$$

$$\int_0^\infty e^{-x \alpha} K_0\left(\sqrt{x^2+\beta^2}\right) dx$$

Nor can it obtain it via the Laplace transform of $K_0\left(\sqrt{x^2+\beta^2}\right)$.

$$\mathcal{L}_x\left[K_0\left(\sqrt{x^2+\beta^2}\right)\right](\alpha)$$

$$\mathcal{L}_x\left[K_0\left(\sqrt{x^2+\beta^2}\right)\right](\alpha)$$

□ Change of variables

Via three changes of variables,

Dt[α] ^= 0; Dt[λ] ^= 0; Dt[y] ^= 0;

Simplify$\left[K_0\left(\lambda \sqrt{x^2+y^2}\right) e^{-\alpha x} \,\mathrm{Dt}[x] \;/.\; \left\{x \to \frac{x}{\alpha},\; y \to \frac{y}{\alpha},\; \lambda \to \lambda \alpha\right\}, \alpha > 0\right]$

$$\frac{e^{-x} K_0\left(\sqrt{x^2+y^2}\,\lambda\right) dx}{\alpha}$$

Simplify$\left[K_0\left(\lambda \sqrt{x^2+y^2}\right) e^{-\alpha x} \,\mathrm{Dt}[x] \;/.\; \left\{x \to \frac{x}{\lambda},\; y \to \frac{y}{\lambda},\; \alpha \to \alpha \lambda\right\}, \lambda > 0\right]$

$$\frac{e^{-x \alpha} K_0\left(\sqrt{x^2+y^2}\right) dx}{\lambda}$$

Simplify$\left[K_0\left(\lambda \sqrt{x^2+y^2}\right) e^{-\alpha x} \,\mathrm{Dt}[x] \;/.\; \left\{x \to x\, y,\; \lambda \to \frac{\lambda}{y},\; \alpha \to \frac{\alpha}{y}\right\}, y > 0\right]$

$$e^{-x \alpha}\, y\, K_0\left(\sqrt{x^2+1}\,\lambda\right) dx$$

we obtain that

$$k(\lambda, \alpha, y) = \frac{1}{\alpha} k(\lambda, 1, y) = \frac{1}{\lambda} k(1, \alpha, y) = y\, k(\lambda, \alpha, 1).$$

■ Numerical integral

For checking we compute the integral numerically.

$$\textbf{NI}[\alpha\_, \beta\_] := \textbf{NIntegrate}\left[K_0\left(\sqrt{\beta^2 + x^2}\right) e^{-\alpha x}, \{x, 0, \infty\}\right]$$

■ Special Cases

□ $\beta = 0$

This integral is computed directly.

$$\textbf{Assuming}\left[\alpha > 0, \int_0^\infty K_0(x)\, e^{-\alpha x}\, dx\right]$$

$$\frac{\cos^{-1}(\alpha)}{\sqrt{1 - \alpha^2}}$$

Alternatively, we obtain it via the Laplace transform.

$$\mathcal{L}_x[K_0(x)](\alpha)$$

$$\frac{\cos^{-1}(\alpha)}{\sqrt{1 - \alpha^2}}$$

$$\textbf{FullSimplify}\left[\frac{\cos^{-1}(\alpha)}{\sqrt{1 - \alpha^2}} = \frac{\cosh^{-1}(\alpha)}{\sqrt{\alpha^2 - 1}}, \alpha > 1\right]$$

True

$$\lim_{\alpha \to 1} \frac{\cos^{-1}(\alpha)}{\sqrt{1 - \alpha^2}}$$

1

□ $\alpha = 0$

The case $\alpha = 0$ can be computed in closed form. *Mathematica* does not do this case directly:

$$\int_0^\infty K_0\left(\sqrt{x^2 + \beta^2}\right) dx$$

$$\int_0^\infty K_0\left(\sqrt{x^2 + \beta^2}\right) dx$$

However, after the change of variables,

$$\textbf{Dt}[\beta]\ \hat{}= 0;$$

$$\textbf{Solve}\left[y == \sqrt{\beta^2 + x^2}, x\right]$$

$$\left\{\left\{x \to -\sqrt{y^2 - \beta^2}\right\}, \left\{x \to \sqrt{y^2 - \beta^2}\right\}\right\}$$

$$\textbf{Simplify}\left[K_0\left(\sqrt{\beta^2+x^2}\right)\frac{\text{Dt}[x]}{\text{Dt}[y]}\ /.\ x\to\sqrt{y^2-\beta^2},\ y>\beta>0\right]$$

$$\frac{y\,K_0(y)}{\sqrt{y^2-\beta^2}}$$

the integral can be evaluated.

$$\textbf{Assuming}\left[\beta>0,\ \int_\beta^\infty\frac{y\,K_0(y)}{\sqrt{y^2-\beta^2}}\,dy\right]$$

$$\frac{e^{-\beta}\,\pi}{2}$$

□ Plot

Utilising the special cases, to avoid problems with the numerical integral in these limits,

$$\text{NI}[0\ |\ 0.,\ \beta\_] = \frac{e^{-\beta}\,\pi}{2};$$

$$\text{NI}[\alpha\_,\ 0\ |\ 0.] = \begin{cases}\frac{\cos^{-1}(\alpha)}{\sqrt{1-\alpha^2}} & 0<\alpha<1\\ \frac{\cosh^{-1}(\alpha)}{\sqrt{\alpha^2-1}} & \alpha>1\\ 1 & \alpha=1\end{cases};$$

here is a plot of the integral for $0 \le \alpha < 3$ and $0 \le \beta < 3$.

```
Plot3D[NI[α, β], {α, 0, 3}, {β, 0, 3},
  PlotRange → All, PlotPoints → 100, Mesh → False]
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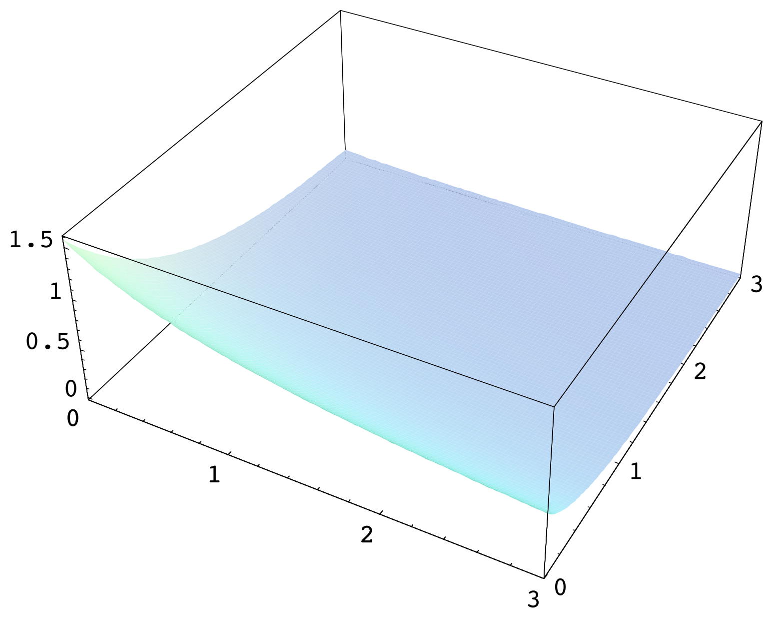

- SurfaceGraphics -

□ $\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) x^n \, dx$

After series expansion of the exponential, the more general integral

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) x^n \, dx$$

can be evaluated.

Simplify$\left[K_0\left(\sqrt{\beta^2+x^2}\right) x^n \dfrac{\text{Dt}[x]}{\text{Dt}[y]} \text{ /. } x \to \sqrt{y^2-\beta^2},\ y>0\right]$

$$y\,(y^2-\beta^2)^{\frac{n-1}{2}} K_0(y)$$

Assuming$\left[\beta > 0 \land n \geq 0, \int_\beta^\infty y\,(y^2-\beta^2)^{\frac{n-1}{2}} K_0(y)\, dy\right]$

$$2^{\frac{n-1}{2}} \beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta)\, \Gamma\left(\frac{n+1}{2}\right)$$

Hence we conclude that

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) x^n \, dx = 2^{\frac{n-1}{2}} \beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta)\, \Gamma\left(\frac{n+1}{2}\right)$$

■ General Case

□ $\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) e^{-\alpha x} \, dx$

Using $e^{-\alpha x} = \sum_{n=0}^\infty (-\alpha x)^n / n!$, we have

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right) e^{-\alpha x} \, dx = \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} 2^{\frac{n-1}{2}} \beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta)\, \Gamma\left(\frac{n+1}{2}\right).$$

$\alpha = 0$ and $\beta = 0$

Here are the low-order terms in the series expansion about $\alpha = 0$.

Simplify /@ Series$\left[\sum_{n=0}^{6} \dfrac{(-\alpha)^n}{n!} 2^{\frac{n-1}{2}} \beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta)\, \Gamma\left(\dfrac{n+1}{2}\right), \{\alpha, 0, 5\}\right]$

$$\frac{e^{-\beta}\pi}{2} - \beta K_1(\beta)\,\alpha + \frac{1}{4} e^{-\beta}\pi(\beta+1)\,\alpha^2 - \frac{1}{3}(\beta^2 K_2(\beta))\,\alpha^3 + \frac{1}{16} e^{-\beta}\pi(\beta^2+3\beta+3)\,\alpha^4 - \frac{1}{15}(\beta^3 K_3(\beta))\,\alpha^5 + O(\alpha^6)$$

Clearly, when $\alpha = 0$, we recover the formula for $G(0, \beta)$.

Normal[%] /. α **-> 0**

$$\frac{e^{-\beta}\pi}{2}$$

Also, when $\beta = 0$, we recover the formula for $G(\alpha, 0)$ since $\beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta) \to 2^{\frac{n-1}{2}}\, \Gamma(\frac{n+1}{2})$ as $\beta \to 0$.

ser = FullSimplify$\left[\text{Series}\left[\beta^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(\beta), \{\beta, 0, 1\}\right], n \geq 0\right]$

$$\left(2^{-\frac{n}{2}-\frac{3}{2}}\, \Gamma\left(\frac{1}{2}(-n-1)\right)\beta + O(\beta^3)\right)\beta^n + \left(2^{\frac{n-1}{2}}\, \Gamma\left(\frac{n+1}{2}\right) + O(\beta^2)\right)$$

Simplify[Normal[ser] /. $\beta \to 0$, $n \geq 0$]

$$2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)$$

$$\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\, 2^{\frac{n-1}{2}}\, \Gamma\left(\frac{n+1}{2}\right) \text{ \% // FullSimplify}$$

$$\frac{\cos^{-1}(\alpha)}{\sqrt{1-\alpha^2}}$$

Numerical Checks

Define the numerical sum.

$$\text{NS}[\alpha\_,\ \beta\_] := \text{NSum}\left[\frac{(-\alpha)^n\, 2^{\frac{n-1}{2}}\, \beta^{\frac{n+1}{2}}\, K_{\frac{n+1}{2}}(\beta)\, \Gamma\left(\frac{n+1}{2}\right)}{n!},\ \{n, 0, \infty\}\right]$$

The numerical sum is convergent for $\alpha < 1$,

With[{α = 0.1, β = 0.2}, {NI[α, β], NS[α, β]}]

{1.197649655, 1.197649655}

With[{α = 0.2, β = 1}, {NI[α, β], NS[α, β]}]

{0.4769453641, 0.4769453641}

But divergent for $\alpha > 1$.

With[{α = 1.01, β = 1.3}, {NI[α, β], NS[α, β]}]

{0.1905574992, ComplexInfinity}

□ Identities

The following identities are implicit in Abramowitz and Stegun 9.1.75-80 (online at www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP),

$$K_0\left(\lambda\sqrt{x^2+y^2}\right) = \sum_{k=-\infty}^{\infty} (-1)^k\, K_{2k}(\lambda x)\, I_{2k}(\lambda y),\ x > y \tag{1}$$

$$K_n\left(\lambda\sqrt{x^2+y^2}\right) T_n\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \sum_{k=-\infty}^{\infty} (-1)^k\, K_{2k+n}(\lambda x)\, I_{2k}(\lambda y),\ x > y \tag{2}$$

$$I_0\left(\lambda\sqrt{x^2+y^2}\right) = \sum_{k=-\infty}^{\infty} (-1)^k\, I_{2k}(\lambda x)\, I_{2k}(\lambda y) \tag{3}$$

$$I_n\left(\lambda\sqrt{x^2+y^2}\right) T_n\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \sum_{k=-\infty}^{\infty} (-1)^k\, I_{2k+n}(\lambda x)\, I_{2k}(\lambda y) \tag{4}$$

and equation 9.6.54, also listed at functions.wolfram.com/03.02.23.0004.01,

$$K_0(z) = -\left(\ln\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + 2\sum_{k=1}^{\infty} \frac{I_{2k}(z)}{k} \tag{5}$$

Series Expansion

As a check, we compute the series expansion in y about $y = 0$ for the first four identities,

$$\text{FullSimplify}\left[\text{Series}\left[K_0\left(\lambda\sqrt{x^2+y^2}\right) - \sum_{k=-4}^{4} (-1)^k\, K_{2k}(\lambda x)\, I_{2k}(\lambda y),\ \{y, 0, 9\}\right],\ x > 0\right]$$

$$O(y^{10})$$

Table[{n, FullSimplify[

$$\text{Series}\left[K_n\left(\lambda\sqrt{x^2+y^2}\right)T_n\left(\frac{x}{\sqrt{x^2+y^2}}\right)-\sum_{k=-4}^{4}(-1)^k K_{2k+n}(\lambda x)\, I_{2k}(\lambda y),\right.$$

{y, 0, 5}], x > 0]}, {n, 0, 3}]

$$\begin{pmatrix} 0 & O(y^6) \\ 1 & O(y^6) \\ 2 & O(y^6) \\ 3 & O(y^6) \end{pmatrix}$$

$$\text{FullSimplify}\left[\text{Series}\left[I_0\left(\lambda\sqrt{x^2+y^2}\right)-\sum_{k=-4}^{4}(-1)^k I_{2k}(\lambda x)\, I_{2k}(\lambda y),\ \{y,0,9\}\right],\ x>0\right]$$

$O(y^{10})$

Table[{n, FullSimplify[

$$\text{Series}\left[I_n\left(\lambda\sqrt{x^2+y^2}\right)T_n\left(\frac{x}{\sqrt{x^2+y^2}}\right)-\sum_{k=-4}^{4}(-1)^k I_{2k+n}(\lambda x)\, I_{2k}(\lambda y),\right.$$

{y, 0, 5}], x > 0]}, {n, 0, 3}]

$$\begin{pmatrix} 0 & O(y^6) \\ 1 & O(y^6) \\ 2 & O(y^6) \\ 3 & O(y^6) \end{pmatrix}$$

and the series expansion in z about $z = 0$ for the third identity.

$$\text{Series}\left[K_0(z)+\left(\ln\left(\frac{z}{2}\right)+\gamma\right)I_0(z)-2\sum_{k=1}^{4}\frac{I_{2k}(z)}{k},\ \{z,0,9\}\right]$$

$O(z^{10})$

Application of (1)

Formally, (1) permits a “separation” of the variables:

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx =$$

$$\sum_{k=-\infty}^{\infty}(-1)^k\left(I_{2k}(\beta)\int_\beta^\infty K_{2k}(x)\,e^{-\alpha x}\,dx + K_{2k}(\beta)\int_0^\beta I_{2k}(x)\,e^{-\alpha x}\,dx\right) =$$

$$I_0(\beta)\int_\beta^\infty K_0(x)\,e^{-\alpha x}\,dx + K_0(\beta)\int_0^\beta I_0(x)\,e^{-\alpha x}\,dx +$$

$$2\sum_{k=1}^{\infty}(-1)^k\left(I_{2k}(\beta)\int_\beta^\infty K_{2k}(x)\,e^{-\alpha x}\,dx + K_{2k}(\beta)\int_0^\beta I_{2k}(x)\,e^{-\alpha x}\,dx\right).$$

However, there does not appear to be a simple closed-form for the resulting integrals.

Defining the (truncated) numerical sum as

NS[1][α_, β_, kmax_: 50] :=

$$\sum_{k=-\text{kmax}}^{\text{kmax}}(-1)^k\,(I_{2k}(\beta)\,\text{NIntegrate}[e^{-\alpha x}K_{2k}(x),\{x,\beta,\infty\}]+$$

$$K_{2k}(\beta)\,\text{NIntegrate}[e^{-\alpha x}I_{2k}(x),\{x,0,\beta\}])$$

or as

NS[1][α_, β_, kmax_: 50] := ($I_0(\beta)$ NIntegrate[$e^{-\alpha x} K_0(x)$, {x, β, ∞}] + $K_0(\beta)$ NIntegrate[$e^{-\alpha x} I_0(x)$, {x, 0, β}]) + $2\sum_{k=1}^{\text{kmax}}(-1)^k$ ($I_{2k}(\beta)$ NIntegrate[$e^{-\alpha x} K_{2k}(x)$, {x, β, ∞}] + $K_{2k}(\beta)$ NIntegrate[$e^{-\alpha x} I_{2k}(x)$, {x, 0, β}])

one obtains reasonable results.

With[{α = 3., β = 1.}, {NI[α, β], NS[1][α, β]}]

{0.126113958, 0.1261188372}

With[{α = 1., β = 3.}, {NI[α, β], NS[1][α, β]}]

{0.0279245832, 0.02793920926}

Application of (5) and (2)

Another approach is to use (5),

$$K_0\left(\sqrt{x^2+\beta^2}\right) = -\left(\frac{1}{2}\ln(x^2+\beta^2)-\ln(2)+\gamma\right)I_0\left(\sqrt{x^2+\beta^2}\right)+2\sum_{k=1}^{\infty}\frac{I_{2k}\left(\sqrt{x^2+\beta^2}\right)}{k},$$

followed by (2)

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx = (\ln(2)-\gamma)\int_0^\infty I_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx - \frac{1}{2}\int_0^\infty \ln(x^2+\beta^2)\,I_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx + 2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\infty I_{2k}\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx.$$

The first integral is

$$\int_0^\infty I_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx = \sum_{k=-\infty}^{\infty}(-1)^k\left(I_{2k}(\beta)\int_0^\infty I_{2k}(x)\,e^{-\alpha x}\,dx\right) = I_0(\beta)\int_0^\infty I_0(x)\,e^{-\alpha x}\,dx + 2\sum_{k=1}^{\infty}(-1)^k\left(I_{2k}(\beta)\int_0^\infty I_{2k}(x)\,e^{-\alpha x}\,dx\right)$$

and the resulting integrals can be computed in closed-form.

Assuming[$\alpha > \lambda > 0 \wedge \nu \geq 0$, $\int_0^\infty I_\nu(\lambda x)\,e^{-\alpha x}\,dx$]

$$\frac{\left(\frac{\lambda}{\alpha+\sqrt{\alpha^2-\lambda^2}}\right)^\nu}{\sqrt{(\alpha-\lambda)(\alpha+\lambda)}}$$

□ Alternative Forms

Considering the even *n* and odd *n* cases separately, one obtains

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)e^{-\alpha x}\,dx = \beta\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha^2\beta}{2}\right)^n\left(\sqrt{\frac{\pi}{2\beta}}\,K_{n+\frac{1}{2}}(\beta)\right)-\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{1}{2}\right)_n}\left(\frac{\alpha^2\beta}{2}\right)^n K_n(\beta), \tag{6}$$

or, equivalently,

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)\cosh(\alpha\,x)\,dx = \beta\sum_{n=0}^\infty \frac{1}{n!}\left(\frac{\alpha^2\,\beta}{2}\right)^n \left(\sqrt{\frac{\pi}{2\,\beta}}\,K_{n+\frac{1}{2}}(\beta)\right), \qquad (7)$$

and

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)\sinh(\alpha\,x)\,dx = \frac{1}{\alpha}\sum_{n=1}^\infty \frac{1}{\left(\frac{1}{2}\right)_n}\left(\frac{\alpha^2\,\beta}{2}\right)^n K_n(\beta), \qquad (8)$$

where the combination $\sqrt{\frac{\pi}{2\beta}}\,K_{n+\frac{1}{2}}(\beta)$ is a modified spherical Bessel function of the third kind:

Simplify[Table[{n, $\beta^n\sqrt{\frac{\pi}{2\,\beta}}\,K_{n+\frac{1}{2}}(\beta)$}, {n, 0, 5}], $\beta > 0$]

$$\begin{pmatrix} 0 & \frac{e^{-\beta}\,\pi}{2\,\beta} \\ 1 & \frac{e^{-\beta}\,\pi\,(\beta+1)}{2\,\beta} \\ 2 & \frac{e^{-\beta}\,\pi\,(\beta^2+3\,\beta+3)}{2\,\beta} \\ 3 & \frac{e^{-\beta}\,\pi\,(\beta^3+6\,\beta^2+15\,\beta+15)}{2\,\beta} \\ 4 & \frac{e^{-\beta}\,\pi\,(\beta^4+10\,\beta^3+45\,\beta^2+105\,\beta+105)}{2\,\beta} \\ 5 & \frac{e^{-\beta}\,\pi\,(\beta^5+15\,\beta^4+105\,\beta^3+420\,\beta^2+945\,\beta+945)}{2\,\beta} \end{pmatrix}$$

I have been able to reduce (7) but not (8).

□ Spherical Bessel Generating functions

Using Abramowitz and Stegun 10.1.40

$$\frac{1}{z}\cos\left(\sqrt{z^2-2\,z\,t}\right) = \sum_{n=0}^\infty \frac{t^n}{n!}\,j_{n-1}(z),$$

one obtains that

$$\Longrightarrow \partial_t\left(\frac{1}{z}\cos\left(\sqrt{z^2-2\,z\,t}\right)\right) = \frac{\sin\left(\sqrt{z^2-2\,t\,z}\right)}{\sqrt{z^2-2\,t\,z}} = \sum_{n=0}^\infty \frac{t^n}{n!}\,j_n(z) = j_0\left(\sqrt{z^2-2\,t\,z}\right).$$

Similarly, subtracting 10.2.30 (with $t \to -i\,t$) from 10.2.31, (with $t \to +i\,t$), we have

$$\frac{1}{z}\exp\left(-\sqrt{z^2-2\,z\,t}\right) = \sum_{n=0}^\infty \frac{(-t)^n}{n!}\sqrt{\frac{\pi}{2\,z}}\left(I_{n-\frac{1}{2}}(z) - I_{-n+\frac{1}{2}}(z)\right),\ 2\,|t| < |z|,$$

and using 10.2.4,

$$\frac{1}{z}\exp\left(-\sqrt{z^2-2\,z\,t}\right) = \frac{2}{\pi}\sum_{n=0}^\infty \frac{t^n}{n!}\sqrt{\frac{\pi}{2\,z}}\,K_{n-\frac{1}{2}}(z),\ 2\,|t| < |z|,$$

we obtain

$$\Longrightarrow \partial_t\left(\frac{1}{z}\exp\left(-\sqrt{z^2-2\,z\,t}\right)\right) = \frac{e^{-\sqrt{z^2-2\,t\,z}}}{\sqrt{z^2-2\,t\,z}} = \frac{2}{\pi}\sum_{n=0}^\infty \frac{t^n}{n!}\sqrt{\frac{\pi}{2\,z}}\,K_{n+\frac{1}{2}}(z),\ 2\,|t| < |z|,$$

Hence, since

Simplify$\left[\left\{\frac{\beta\, e^{-\sqrt{z^2-2tz}}}{\sqrt{z^2-2tz}}\frac{\pi}{2}, 2|t|<|z|\right\} /. \left\{t \to \frac{\alpha^2\beta}{2}, z\to\beta\right\}, \beta>0\right]$

$\left\{\frac{e^{-\sqrt{1-\alpha^2}\,\beta}\,\pi}{2\sqrt{1-\alpha^2}}, |\alpha|^2<1\right\}$

we have that

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)\cosh(\alpha x)\,dx = \frac{\pi}{2}\frac{e^{-\beta\sqrt{1-\alpha^2}}}{\sqrt{1-\alpha^2}},\ |\alpha|<1.$$

Checks

With$\left[\{\alpha = 0.3, \beta = 3\},\right.$

$\left.\left\{\mathbf{NIntegrate}\left[K_0\left(\sqrt{x^2+\beta^2}\right)\cosh(\alpha x), \{x, 0, \infty\}\right], \frac{\pi\, e^{-\beta\sqrt{1-\alpha^2}}}{2\sqrt{1-\alpha^2}}\right\}\right]$

{0.09412989779, 0.09412989779}

With$\left[\{\alpha = 0.1, \beta = 5\},\right.$

$\left.\left\{\mathbf{NIntegrate}\left[K_0\left(\sqrt{x^2+\beta^2}\right)\cosh(\alpha x), \{x, 0, \infty\}\right], \frac{\pi\, e^{-\beta\sqrt{1-\alpha^2}}}{2\sqrt{1-\alpha^2}}\right\}\right]$

{0.01090723103, 0.01090723103}

Simplify$\Big[$

$\mathbf{Series}\left[\beta\sum_{n=0}^{5}\frac{1}{n!}\left(\frac{\alpha^2\beta}{2}\right)^n\left(\sqrt{\frac{\pi}{2\beta}}\,K_{n+\frac{1}{2}}(\beta)\right)-\frac{\pi}{2}\frac{e^{-\beta\sqrt{1-\alpha^2}}}{\sqrt{1-\alpha^2}}, \{\alpha, 0, 11\}\right], \beta>0\Big]$

$O(\alpha^{12})$

Extension

Putting $\alpha \to i\,\alpha$, one deduces that

$$\int_0^\infty K_0\left(\sqrt{x^2+\beta^2}\right)\cos(\alpha x)\,dx = \frac{\pi}{2}\frac{e^{-\beta\sqrt{\alpha^2+1}}}{\sqrt{\alpha^2+1}}$$

Checks

With$\left[\{\alpha = 0.5, \beta = 3\},\right.$

$\left.\left\{\mathbf{NIntegrate}\left[K_0\left(\sqrt{x^2+\beta^2}\right)\cos(\alpha x), \{x, 0, \infty\}\right], \frac{\pi\, e^{-\beta\sqrt{\alpha^2+1}}}{2\sqrt{\alpha^2+1}}\right\}\right]$

{0.04909043662, 0.04909043662}

With$\left[\{\alpha = 5., \beta = 2\},\right.$

$\left\{\mathbf{NIntegrate}\left[K_0\left(\sqrt{x^2+\beta^2}\right)\cos(\alpha x), \{x, 0, \infty\}, \mathbf{Method} \to \mathbf{Oscillatory}\right],\right.$

$\left.\left.\frac{\pi\, e^{-\beta\sqrt{\alpha^2+1}}}{2\sqrt{\alpha^2+1}}\right\}\right]$

{0.0000114731092, 0.0000114731092}