Here is the integral that you posted.

$$\rho_{11}\ (0,\, T) = \lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\frac{1}{2\,\pi\,i}\int_{\delta+0}^{T-\delta}\left(\frac{i}{\sqrt{\eta - i\,\epsilon}\,\left(\sqrt{\eta - i\,\epsilon}\;i\,\kappa + 1\right)} - \frac{i}{\sqrt{i\,\epsilon + \eta}\,\left(\sqrt{i\,\epsilon + \eta}\;i\,\kappa + 1\right)}\right)d\eta$$

Defining the integrand, adding the two terms instead of subtracting them,

$$\textbf{int}[\epsilon\_,\,\kappa\_][\eta\_] = \frac{i}{\sqrt{\eta - i\,\epsilon}\,\left(i\,\sqrt{\eta - i\,\epsilon}\;\kappa + 1\right)} + \frac{i}{\sqrt{i\,\epsilon + \eta}\,\left(i\,\sqrt{i\,\epsilon + \eta}\;\kappa + 1\right)};$$

then the real part of the integrand, for $\epsilon = 0$ is

$$\textbf{Simplify}\left[\textbf{ComplexExpand}\left[\textbf{Re}\left[\frac{1}{2\,\pi\,i}\ \textbf{int}[0,\,\kappa][\eta]\right],\ \textbf{TargetFunctions} \to \{\textbf{Re},\,\textbf{Im}\}\right],\ \epsilon > 0 \land \eta > 0\right]$$

$$\frac{1}{\sqrt{\eta}\ (\pi\,\eta\,\kappa^2 + \pi)}$$

Computing the integral of this expression is straightforward.

$$\textbf{reans}[\kappa\_,\,\textbf{T\_}] = \textbf{Assuming}\left[\kappa \neq 0 \land T > 0 \land \{\kappa,\,T\} \in \mathbb{R},\ \frac{1}{\pi}\int_0^T \frac{1}{\sqrt{\eta}\ (\eta\,\kappa^2 + 1)}\,d\eta\right]$$

$$\frac{2\tan^{-1}\left(\sqrt{T}\ |\kappa|\right)}{\pi\,|\kappa|}$$

This result simplifies further.

Simplify[reans[κ, T], $\kappa > 0$]

$$\frac{2\tan^{-1}\left(\sqrt{T}\ \kappa\right)}{\pi\,\kappa}$$

Simplify[reans[κ, T], $\kappa < 0$]

$$\frac{2\tan^{-1}\left(\sqrt{T}\ \kappa\right)}{\pi\,\kappa}$$

Hence we write

$$\textbf{reans}[\kappa\_,\,\textbf{T\_}] = \frac{2\tan^{-1}\left(\sqrt{T}\ \kappa\right)}{\pi\,\kappa};$$

The full integrand for $\epsilon = 0$ is

$$\textbf{Simplify}\left[\textbf{ComplexExpand}\left[\frac{1}{2\,\pi\,i}\ \textbf{int}[0,\,\kappa][\eta],\ \textbf{TargetFunctions} \to \{\textbf{Re},\,\textbf{Im}\}\right],\ \epsilon > 0 \land \eta > 0\right]$$

$$-\frac{i}{\pi\,\eta\,\kappa - i\,\pi\,\sqrt{\eta}}$$

Integration of this expression is immediate.

$$\text{ans}[\kappa\_, \text{T}\_] = \int_0^T \% \, d\eta$$

$$-\frac{2\, i \log\left(i \sqrt{T}\, \kappa + 1\right)}{\pi \kappa}$$

The real part of this result agrees with the result obtained above.

FullSimplify[ComplexExpand[Re[ans[κ, *T*]], TargetFunctions → {Re, Im}], κ ≠ 0 ∧ *T* > 0 ∧ {κ, *T*} ∈ ℝ]

$$\frac{2 \tan^{-1}\left(\sqrt{T}\, \kappa\right)}{\pi \kappa}$$

Here is the imaginary part of the full result.

imans[κ_, T_] =
FullSimplify[ComplexExpand[Im[ans[κ, *T*]], TargetFunctions → {Re, Im}], κ ≠ 0 ∧ *T* > 0 ∧ {κ, *T*} ∈ ℝ]

$$-\frac{\log(T \kappa^2 + 1)}{\pi \kappa}$$

Note that one can simply use the antiderivative of the integrand directly, yielding and equivalent result.

$$\int \text{int}[\epsilon, \kappa][\eta] \, d\eta$$

$$i \left(-\frac{2\, i \log(\sqrt{\eta - i \epsilon}\, \kappa - i)}{\kappa} - \frac{2\, i \log(\sqrt{i \epsilon + \eta}\, \kappa - i)}{\kappa}\right)$$

$$\frac{\%}{2 \pi i} \; /. \; \epsilon \to 0$$

$$-\frac{2\, i \log(\sqrt{\eta}\, \kappa - i)}{\pi \kappa}$$

$$(\% \; /. \; \eta \to T) - (\% \; /. \; \eta \to 0)$$

$$\frac{1}{\kappa} - \frac{2\, i \log\left(\sqrt{T}\, \kappa - i\right)}{\pi \kappa}$$

Numerical check

Here we check the integral numerically for small ϵ with $\kappa = 1$ and $T = 2$.

$$\frac{1}{2 \pi i} \text{NIntegrate}[\text{int}[10^{-6}, 1][\eta], \{\eta, 0, 2\}]$$

0.6077232905 – 0.3496991526 i

$$\frac{1}{2 \pi i} \text{NIntegrate}[\text{int}[10^{-15}, 1][\eta], \{\eta, 0, 2\}]$$

0.6081734436 – 0.3496991526 i

Here is the first closed form

ans[1, 2.]

0.608173448 − 0.3496991526 i

and here is the second.

$$\frac{1}{\kappa} - \frac{2\,i\,\log\left(\sqrt{T}\ \kappa - i\right)}{\pi\,\kappa}\ /.\ \{T \to 2,\ \kappa \to 1.\}$$

0.608173448 − 0.3496991526 i

Here is the real part of the integral,

reans[1, 2.]

0.608173448

and here is the imaginary part.

imans[1, 2.]

−0.3496991526