Appell Functions

At mathworld.wolfram.com/AppellHypergeometricFunction.html, it says

> Appell defined the functions in 1880, and Picard showed in 1881 that they may all be expressed by integrals of the form
>
> $$\int_0^1 u^\alpha (1-u)^\beta (1-x\,u)^\gamma (1-y\,u)^\delta \, du \qquad (5)$$
>
> (Bailey 1934, pp. 76-79).

However, my reading of Bailey does not agree with this statement. On pages 76-77 he states that the functions F_1, F_2, F_3 can be expressed in terms of double integrals:

$$\frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(c-b_1-b_2)}{\Gamma(c)} F_1\,(a; b_1, b_2; c; x, y) = \int_0^1 \int_0^{1-v} u^{b_1-1}\, v^{b_2-1}\, (1-u-v)^{c-b_1-b_2-1}\, (1-u\,x-v\,y)^{-a}\, (1-y\,u)^\delta \, du\, dv, \qquad (1)$$

$$\frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(c_1-b_1)\,\Gamma(c_2-b_2)}{\Gamma(c_1)\,\Gamma(c_2)} F_2(a; b_1, b_2; c_1, c_2; x, y) = \int_0^1 \int_0^1 u^{b_1-1}\, v^{b_2-1}\, (1-u)^{c_1-b_1-1}\, (1-v)^{c_2-b_1-1}\, (1-u\,x-v\,y)^{-a}\, du\, dv, \qquad (2)$$

$$\frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(c-b_1-b_2)}{\Gamma(c)} F_3(a_1, a_2; b_1, b_2; c; x, y) = \int_0^1 \int_0^{1-v} u^{b_1-1}\, v^{b_2-1}\, (1-u-v)^{c-b_1-b_2-1}\, (1-u\,x)^{-a_1}\, (1-v\,y)^{-a_2}\, du\, dv. \qquad (3)$$

He goes on to state that

> There appears to be no simple integral representation of this type for the function F_4.

and then

> The function F_1 can also be expressed by a simple integral, the formula being
>
> $$\frac{\Gamma(a)\,\Gamma(c-a)}{\Gamma(c)} F_1\,(a; b_1, b_2; c; x, y) = \int_0^1 u^{a-1}\, (1-u)^{c-a-1}\, (1-u\,x)^{-b_1}\, (1-u\,y)^{-b_2}\, du. \qquad (4)$$

followed by

> The four functions can also be expressed as double contour integrals taken along contours of the Barnes' type.

so it is not correct to state that they may all be expressed by integrals of the form (5).

Later on it says that

> $F_1\,(\alpha; \beta, \beta'; \gamma; x, y)$ reduces to the hypergeometric function in the cases … (10) and (11).

To me, this implies that these are the only reductions. Two other reductions are given in Bailey:

$$F_1\,(\alpha;\beta,\beta';\gamma;x,x)=(1-x)^{\gamma-\alpha-\beta-\beta'}\;{}_2F_1(\gamma-\alpha,-\beta+\gamma-\beta';\gamma;x)$$
$$={}_2F_1(\alpha,\beta+\beta';\gamma;x)$$
$$F_1\,(\alpha;\beta,\beta';\beta+\beta';x,y)=(1-y)^{-\alpha}\;{}_2F_1\Big(\alpha,\beta;\beta+\beta';\frac{x-y}{1-y}\Big)$$

■ Notation

■ $F_2\,(a;b_1,b_2;c_1,c_2;x,y)$

□ Summation

$$F_2(a;b_1,b_2;c_1,c_2;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{m!\,n!\,(c_1)_m\,(c_2)_n}\,x^m\,y^n \tag{5}$$

Direct implementation via truncating this sum (**NSum** is very slow):

AppellF2S[a_, {b1_, b2_}, {c1_, c2_}, {x_, y_}, mmax_: 50, nmax_: 50] :=

$$\sum_{m=0}^{\text{mmax}}\sum_{n=0}^{\text{nmax}}\frac{(a)_{m+n}\,(\text{b1})_m\,(\text{b2})_n}{m!\,n!\,(\text{c1})_m\,(\text{c2})_n}\,x^m\,y^n$$

The sum is rapidly convergent for some parameters,

With[{a = 1, b_1 = 1, b_2 = 1, c_1 = 3, c_2 = 3, x = 0.5, y = 0.3},
AppellF2S[a, {b_1, b_2}, {c_1, c_2}, {x, y}]]

1.422020759

but, not suprisingly, is divergent for others.

With[{a = 1, b_1 = 1, b_2 = 1, c_1 = 3, c_2 = 3, x = −5., y = i},
AppellF2S[a, {b_1, b_2}, {c_1, c_2}, {x, y}]]

$-3.635980984\times10^{57}+1.896463615\times10^{57}\,i$

□ Transformations

For a range of parameters, we can use the transformations

$$\ldots_1,b_2;c_1,c_2;x,y)=(1-x)^{-a}\,F_2\Big(a;c_1-b_1,b_2;c_1,c_2;\frac{-x}{1-x},\frac{y}{1-x}\Big)$$
$$=(1-y)^{-a}\,F_2\Big(a;b_1,c_2-b_2;c_1,c_2;\frac{x}{1-y},-\frac{y}{1-y}\Big) \tag{6}$$
$$=(1-x-y)^{-a}\,F_2\Big(a;c_1-b_1,c_2-b_2;c_1,c_2;-\frac{x}{1-x-y},-\frac{}{1}$$

to obtain convergent sums. For the above example, using the first transformation we obtain

With[{a = 1, b_1 = 1, b_2 = 1, c_1 = 3, c_2 = 3, x = −5, y = i},
$(1-x)^{-a}$ AppellF2[a, {c_1-b_1, b_2}, {c_1, c_2}, {$\frac{-x}{1-x}$, $\frac{y}{1-x}$}]]

$$\frac{1}{6}\,F_2\Big(1,\{2,1\},\{3,3\},\Big\{\frac{5}{6},\frac{i}{6}\Big\}\Big)$$

which evaluates to

N[%] /. AppellF2 → AppellF2S

0.4369116872 + 0.07610506571 i

Using the third transformation we obtain

$$\mathbf{With}\Big[\{a = 1,\ b_1 = 1,\ b_2 = 1,\ c_1 = 3,\ c_2 = 3,\ x = -5,\ y = i\},\ (1 - x - y)^{-a}\ \mathbf{AppellF2}\Big[a,\ \{c_1 - b_1,\ c_2 - b_2\},\ \{c_1,\ c_2\},\ \Big\{-\frac{x}{1 - x - y},\ -\frac{y}{1 - x - y}\Big\}\Big]\Big]$$

$$\left(\frac{6}{37} + \frac{i}{37}\right) F_2\Big(1,\ \{2,\ 2\},\ \{3,\ 3\},\ \Big\{\frac{30}{37} + \frac{5\,i}{37},\ \frac{1}{37} - \frac{6\,i}{37}\Big\}\Big)$$

which evaluates to

N[%] /. AppellF2 → AppellF2S

0.4369116872 + 0.07610506571 i

Two special cases can be used as tests.

$$F_2\,(a;\,b_1,\,b_2;\,c,\,a;\,x,\,y) = (1-y)^{-b_2}\ F_1\left(b_1;\,a-b_2,\,b_2;\,c;\,x,\,\frac{x}{1-y}\right) \tag{7}$$

Here we compute $F_2\,(1;\,1,\,2;\,3,\,1;\,-0.5,\,0.3 + 0.2\,i)$ three ways. Here is the direct sum.

```
With[{a = 1, b1 = 1, b2 = 2, c = 3, x = −0.5, y = 0.3 + 0.2 i},
  AppellF2S[a, {b1, b2}, {c, a}, {x, y}]]
```

1.342015306 + 0.6810484535 i

```
With[{a = 1, b1 = 1, b2 = 2, c = 3, x = −0.5, y = 0.3 + 0.2 i},
  AppellF2S[a, {b2, b1}, {a, c}, {y, x}]]
```

1.342015306 + 0.6810484535 i

Here we use (6).

$$\mathbf{With}\Big[\{a = 1,\ b_1 = 1,\ b_2 = 2,\ c = 3,\ x = -0.5,\ y = 0.3 + 0.2\,i\},\ (1 - x)^{-a}\ \mathbf{AppellF2S}\Big[a,\ \{c - b_1,\ b_2\},\ \{c,\ a\},\ \Big\{\frac{-x}{1 - x},\ \frac{y}{1 - x}\Big\}\Big]\Big]\ //\ N$$

1.342015306 + 0.6810484534 i

Here we use (7).

$$\mathbf{With}\Big[\{a = 1,\ b_1 = 1,\ b_2 = 2,\ c = 3,\ x = -0.5,\ y = 0.3 + 0.2\,i\},\ (1 - y)^{-b_2}\ F_1\left(b_1;\ a - b_2,\ b_2;\ c;\ x,\ \frac{x}{1 - y}\right)\Big]\ //\ \mathbf{Expand}$$

1.342015306 + 0.6810484534 i

Here is the second special case.

$$F_2\,(a;\,b_1,\,b_2;\,a,\,a;\,x,\,y) = (1-x)^{-b_1}\ (1-y)^{-b_2}\ {}_2F_1\left(b_1,\,b_2;\,a;\,\frac{x\,y}{(1-x)\,(1-y)}\right) \tag{8}$$

```
With[{a = 1, b1 = 1, b2 = 2, x = −0.2, y = 0.3 + 0.4 i},
  AppellF2S[a, {b1, b2}, {a, a}, {x, y}]]
```

0.8289935168 + 0.9182697417 i

$$\mathbf{With}\Big[\{a = 1,\ b_1 = 1,\ b_2 = 2,\ x = -0.2,\ y = 0.3 + 0.4\,i\},\ (1 - x)^{-b_1}\ (1 - y)^{-b_2}\ {}_2F_1\left(b_1,\ b_2;\ a;\ \frac{x\,y}{(1 - x)\,(1 - y)}\right)\Big]$$

0.8289935168 + 0.9182697417 i

□ Integral

Direct implementation of (2) via numerical integration:

$$\textbf{AppellF2I1[a\_, \{b1\_, b2\_\}, \{c1\_, c2\_\}, \{x\_, y\_\}] :=} \frac{\Gamma(c1)\,\Gamma(c2)}{\Gamma(b1)\,\Gamma(b2)\,\Gamma(c1-b1)\,\Gamma(c2-b2)} \text{NIntegrate}\left[\frac{u^{b1-1}\,v^{b2-1}\,(1-u)^{c1-b1-1}\,(1-v)^{c2-b1-1}}{(1-u\,x-v\,y)^{a}}, \{v,0,1\}, \{u,0,1\}\right]$$

**With[{$a = 1$, $b_1 = 1$, $b_2 = 1$, $c_1 = 3$, $c_2 = 3$, $x = 0.5$, $y = 0.3$},
AppellF2I1[a, {b_1, b_2}, {c_1, c_2}, {x, y}]]**

1.422020758

**With[{$a = 1$, $b_1 = 1$, $b_2 = 1$, $c_1 = 3$, $c_2 = 3$, $x = -5$, $y = i$},
AppellF2I1[a, {b_1, b_2}, {c_1, c_2}, {x, y}]]**

0.4369117817 + 0.07610573827 i

A second option is to compute the one integral in closed form, say

$$\int_0^1 u^{b_1-1}\,v^{b_2-1}\,(1-u)^{c_1-b_1-1}\,(1-v)^{c_2-b_1-1}\,(1-u\,x-v\,y)^{-a}\,du$$

$$(1-v)^{-b_1+c_2-1}\,v^{b_2-1}\,\text{If}\Big[\text{Re}(b_1)>0 \bigwedge \text{Re}(v\,y)<1 \bigwedge \text{Re}(b_1)<\text{Re}(c_1) \bigwedge \left(\text{Re}\left(\frac{1-v\,y}{x}\right)\le 0 \bigvee \text{Re}\left(\frac{1-v\,y}{x}\right)\ge 1 \bigvee \text{Im}\left(\frac{1-v\,y}{x}\right)\ne 0\right),$$
$$(1-v\,y)^{-a}\,\Gamma(b_1)\,\Gamma(c_1-b_1)\,{}_2\tilde{F}_1\left(a, b_1; c_1; \frac{x}{1-v\,y}\right),$$
$$\text{Integrate}\Big[(1-u)^{-b_1+c_1-1}\,u^{b_1-1}\,(-u\,x-v\,y+1)^{-a}, \{u,0,1\},$$
$$\text{Assumptions}\to\left(\text{Im}\left(\frac{1-v\,y}{x}\right)=0 \bigwedge 0<\text{Re}\left(\frac{1-v\,y}{x}\right)<1\right) \bigvee \text{Re}(v\,y)\ge 1 \bigvee \text{Re}(b_1)\ge\text{Re}(c_1) \bigvee \text{Re}(b_1)\le 0\Big]\Big]$$

Hence

$$F_2(a; b_1, b_2; c_1, c_2; x, y) = \frac{\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(b_2)\,\Gamma(c_2-b_2)} \int_0^1 (1-v)^{c_2-b_1-1}\,v^{b_2-1}\,(1-v\,y)^{-a}\;{}_2\tilde{F}_1\left(a, b_1; c_1; \frac{x}{1-v\,y}\right) dv, \tag{9}$$

that is

$$\textbf{AppellF2I2[a\_, \{b1\_, b2\_\}, \{c1\_, c2\_\}, \{x\_, y\_\}] :=} \frac{\Gamma(c1)\,\Gamma(c2)}{\Gamma(b2)\,\Gamma(c2-b2)} \text{NIntegrate}\left[(1-v)^{c2-b1-1}\,v^{b2-1}\,(1-v\,y)^{-a}\;{}_2\tilde{F}_1\left(a, b1; c1; \frac{x}{1-v\,y}\right), \{v,0,1\}\right]$$

when the following condition is satisfied:

$$\textbf{test[a\_, \{b1\_, b2\_\}, \{c1\_, c2\_\}, \{x\_, y\_\}] :=} \text{Reduce}\Big[0<v<1 \bigwedge \text{Re(b1)}>0 \bigwedge \text{Re}(v\,y)<1 \bigwedge \text{Re(b1)}<\text{Re(c1)} \bigwedge \left(\text{Re}\left(\frac{1-v\,y}{x}\right)\le 0 \bigvee \text{Re}\left(\frac{1-v\,y}{x}\right)\ge 1 \bigvee \text{Im}\left(\frac{1-v\,y}{x}\right)\ne 0\right)\Big]$$

For the first set of parameters,

test[1, {1, 1}, {3, 3}, {1/2, 3/10}]

$0 < v < 1$

we obtain

```
AppellF2I2[1, {1, 1}, {3, 3}, {1/2, 3/10}]
```

1.422020759

For the second set of parameters,

```
With[{a = 1, b1 = 1, b2 = 1, c1 = 3, c2 = 3, x = -5, y = i},
  {test[a, {b1, b2}, {c1, c2}, {x, y}], AppellF2I2[a, {b1, b2}, {c1, c2}, {x, y}]}]
```

$\{0 < v < 1, 0.4369117887 + 0.07610573909\, i\}$

■ $F_3(a_1, a_2, ; b_1, b_2; c; x, y)$

□ Summation

$$F_3(a_1, a_2; b_1, b_2; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{m!\, n!\, (c)_{m+n}} x^m y^n \tag{10}$$

AppellF3S[{a1_, a2_}, {b1_, b2_}, c_, {x_, y_}, mmax_: 50, nmax_: 50] :=

$$\sum_{m=0}^{\text{mmax}} \sum_{n=0}^{\text{nmax}} \frac{(\text{a1})_m (\text{a2})_n (\text{b1})_m (\text{b2})_n}{m!\, n!\, (c)_{m+n}} x^m y^n$$

AppellF3S[{1, 1}, {2, 2}, 6, {0.5, 0.3}]

1.350080802

□ Integral

Compute the inner integral under a particular set of assumptions.

$$\text{Assuming}\left[v < 1 \bigwedge \text{Re}(b_1) > 0 \bigwedge \text{Re}(c) > \text{Re}(b_1) + \text{Re}(b_2) \bigwedge v + \text{Re}\left(\frac{1}{x}\right) \geq 1, \int_0^{1-v} u^{b_1-1} v^{b_2-1} (1-u-v)^{c-b_1-b_2-1} (1-u\,x)^{-a_1} (1-v\,y)^{-a_2}\, du\right]$$

$$(1-v)^{c-b_2-1} v^{b_2-1} (1-v\,y)^{-a_2}\, \Gamma(b_1)\,\Gamma(c-b_1-b_2)\, {}_2\tilde{F}_1(a_1, b_1; c-b_2; x - v\,x)$$

Hence

$$F_3(a_1, a_2, ; b_1, b_2; c; x, y) = \frac{\Gamma(c)}{\Gamma(b_2)} \int_0^1 (1-v)^{c-b_2-1} v^{b_2-1} (1-v\,y)^{-a_2}\, {}_2\tilde{F}_1(a_1, b_1; c-b_2; x(1-v))\, dv \tag{11}$$

$$\text{AppellF3I[\{a1\_, a2\_\}, \{b1\_, b2\_\}, c\_, \{x\_, y\_\}] :=} \frac{\Gamma(c)}{\Gamma(\text{b2})}$$
$$\text{NIntegrate}\left[(1-v)^{c-\text{b2}-1} v^{\text{b2}-1} (1-v\,y)^{-\text{a2}}\, {}_2\tilde{F}_1(\text{a1}, \text{b1}; c-\text{b2}; x(1-v)), \{v, 0, 1\}\right]$$

AppellF3I[{1, 1}, {2, 2}, 6, {0.5, 0.3}]

1.350080802