■ Symbolic Vector Analysis Using *Mathematica* (Demo)

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In[1]:= **<< ~/mathematica/GeneralVectorAnalysis.m**

Conventional vector analysis notation is built in.

In[2]:= **FullForm/@{ $a\cdot(b+c)$, $(a+d)\times(b\times c)$}**

Out[2]= {DotProduct[a, Plus[b, c]], CrossProduct[Plus[a, d], CrossProduct[b, c]]}

In[3]:= **FullForm/@{$\nabla\times(a+a\times(b\times c))$, $\nabla\cdot(a+b\times d)$}**

Out[3]= {Curl[Plus[a, CrossProduct[a, CrossProduct[b, c]]]], Div[Plus[a, CrossProduct[b, d]]]}

In[4]:= **FullForm/@{∇^2 (f g), $\hat{A}$, |B|, $\mathcal{J}$[r]}**

Out[4]= {Laplacian[Times[f, g]], UnitVector[A], AbsoluteValue[B], Jacobian[r]}

Vector calculations independent of coordinate system.

In[5]:= **DeclareVector[A, B_0, B, C, D, E, F, G, H, J, ξ, Q]; DeclareScalar[a, b, c, d, e, f, g, h]**

In[6]:= **{$A\times A$, $A\cdot(A\times C)$, $\nabla\times(\nabla f)$, $\nabla\cdot(\nabla\times A)$, VectorExpand[$A\times(B\times C)$]}**

Out[6]= $\{0, 0, 0, 0, -C\, A\cdot B+B\, A\cdot C\}$

In[7]:= **VectorExpand/@{$A\times(B\times C)+B\times(C\times A)+C\times(A\times B)$, $(A\times B)\times(C\times D)$, $\nabla\cdot(f\,(A\times B+\nabla\times C))$}**

Out[7]= $\{0, -D\, A\cdot(B\times C)+C\, A\cdot(B\times D), -f\, A\cdot(\nabla\times B)+f\, B\cdot(\nabla\times A)+(A\times B)\cdot(\nabla f)+(\nabla\times C)\cdot(\nabla f)\}$

An example from ideal megnetohydrodynamics

In[8]:= **$\nabla\cdot(B_0)=0$; $Q=\nabla\times(\xi\times B_0)$; $J=\nabla\times Q$**

Out[8]= $\nabla\times(\nabla\times(\xi\times(B_0)))$

In[9]:= **VectorExpand[J]**

Out[9]= $-(\nabla(\nabla\cdot\xi))\times(B_0)-\nabla\times((\xi\cdot\nabla)(B_0))+\nabla\times(((B_0)\cdot\nabla)\xi)-\nabla\times(B_0)\nabla\cdot\xi$

Vector analysis in any well−defined coordinate system.

In[10]:=
```
SetCoordinateSystem["Cartesian"][x, y, z]; A = DefineVector[1, y z, −x, x^2]; B = (∇ × A)[[1]];
SetCoordinateSystem["None"]; Needs["Graphics`PlotField3D`"]
```

In[11]:=
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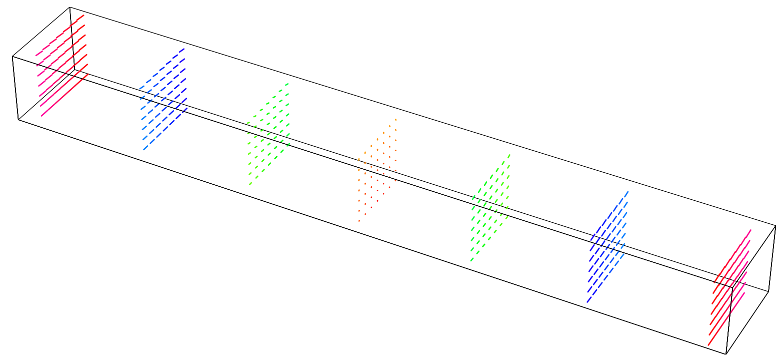
PlotVectorField3D[B, {x, −10, 10}, {y, −1, 1}, {z, −1, 1}, ColorFunction −> Hue];
```



In[12]:=
```
SetCoordinateSystem["Cylindrical"][r, θ, z];
```

In[13]:=
```
A = DefineVector[1, Ar[r, θ, z], Aθ[r, θ, z], Az[r, θ, z]]; B = DefineVector[1, Br[r, θ, z], Bθ[r, θ, z], Bz[r, θ, z]];
```

In[14]:=
```
A · B
```

Out[14]=
$$\mathrm{Ar}[r,\theta,z]\,\mathrm{Br}[r,\theta,z]+\mathrm{Az}[r,\theta,z]\,\mathrm{Bz}[r,\theta,z]+\frac{\mathrm{A}\theta[r,\theta,z]\,\mathrm{B}\theta[r,\theta,z]}{r^2}$$

In[15]:=
```
A × B
```

Out[15]=
$$\mathrm{Vector}\Big[\Big\{r\left(\frac{\mathrm{A}\theta[r,\theta,z]\,\mathrm{Bz}[r,\theta,z]}{r^2}-\frac{\mathrm{Az}[r,\theta,z]\,\mathrm{B}\theta[r,\theta,z]}{r^2}\right),\ r\,(\mathrm{Az}[r,\theta,z]\,\mathrm{Br}[r,\theta,z]-\mathrm{Ar}[r,\theta,z]\,\mathrm{Bz}[r,\theta,z]),\ r\left(-\frac{\mathrm{A}\theta[r,\theta,z]\,\mathrm{Br}[r,\theta,z]}{r^2}+\frac{\mathrm{Ar}[r,\theta,z]\,\mathrm{B}\theta[r,\theta,z]}{r^2}\right)\Big\},$$
$$\Big\{\frac{\mathrm{A}\theta[r,\theta,z]\,\mathrm{Bz}[r,\theta,z]-\mathrm{Az}[r,\theta,z]\,\mathrm{B}\theta[r,\theta,z]}{r},\ \frac{\mathrm{Az}[r,\theta,z]\,\mathrm{Br}[r,\theta,z]-\mathrm{Ar}[r,\theta,z]\,\mathrm{Bz}[r,\theta,z]}{r},\ \frac{-\mathrm{A}\theta[r,\theta,z]\,\mathrm{Br}[r,\theta,z]+\mathrm{Ar}[r,\theta,z]\,\mathrm{B}\theta[r,\theta,z]}{r}\Big\}\Big]$$

In[16]:=
```
∇ · A
```

Out[16]=
$$\frac{1}{r}\left(\mathrm{Ar}[r,\theta,z]+r\,\mathrm{Az}^{(0,0,1)}[r,\theta,z]+\frac{\mathrm{A}\theta^{(0,1,0)}[r,\theta,z]}{r}+r\,\mathrm{Ar}^{(1,0,0)}[r,\theta,z]\right)$$

In[17]:=
```
∇ × A
```

Out[17]=
$$\mathrm{Vector}\Big[\Big\{\frac{-\mathrm{A}\theta^{(0,0,1)}[r,\theta,z]+\mathrm{Az}^{(0,1,0)}[r,\theta,z]}{r},\ r\left(\mathrm{Ar}^{(0,0,1)}[r,\theta,z]-\mathrm{Az}^{(1,0,0)}[r,\theta,z]\right),\ \frac{-\mathrm{Ar}^{(0,1,0)}[r,\theta,z]+\mathrm{A}\theta^{(1,0,0)}[r,\theta,z]}{r}\Big\},$$
$$\Big\{\frac{-\mathrm{A}\theta^{(0,0,1)}[r,\theta,z]+\mathrm{Az}^{(0,1,0)}[r,\theta,z]}{r},\ \frac{\mathrm{Ar}^{(0,0,1)}[r,\theta,z]-\mathrm{Az}^{(1,0,0)}[r,\theta,z]}{r},\ \frac{-\mathrm{Ar}^{(0,1,0)}[r,\theta,z]+\mathrm{A}\theta^{(1,0,0)}[r,\theta,z]}{r}\Big\}\Big]$$

In[18]:= $\{\{A^1, A^2, A^3\}, \{B_1, B_2, B_3\}\}$

Out[18]= $\left\{\left\{\text{Ar}[r,\theta,z], \frac{\text{A}\theta[r,\theta,z]}{r^2}, \text{Az}[r,\theta,z]\right\}, \{\text{Br}[r,\theta,z], \text{B}\theta[r,\theta,z], \text{Bz}[r,\theta,z]\}\right\}$

In[19]:= $|A|$

Out[19]= $\sqrt{\text{Ar}[r,\theta,z]^2 + \text{Az}[r,\theta,z]^2 + \frac{\text{A}\theta[r,\theta,z]^2}{r^2}}$

In[20]:= $\hat{A}$

Out[20]= $\text{Vector}\left[\left\{\frac{\text{Ar}[r,\theta,z]}{\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}, \frac{\text{A}\theta[r,\theta,z]}{\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}, \frac{\text{Az}[r,\theta,z]}{\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}\right\}, \left\{\frac{\text{Ar}[r,\theta,z]}{\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}, \frac{\text{A}\theta[r,\theta,z]}{r^2\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}, \frac{\text{Az}[r,\theta,z]}{\sqrt{\text{Ar}[r,\theta,z]^2+\text{Az}[r,\theta,z]^2+\frac{\text{A}\theta[r,\theta,z]^2}{r^2}}}\right\}\right]$

❆ Coordinate systems for tokamaks are built in. Asymptotic vector analysis is available.

In[21]:= SetCoordinateSystem["TKCircular2", R_0, ϵ][r, θ, φ];

In[22]:= Clear[B]; $B[r,\theta,\varphi]$ = DefineVector$\left[1, 0, \text{Series}\left[\frac{\epsilon\, r^2\, B_0}{R_0\; q[r]\left(1+\frac{\epsilon r}{R_0}\text{Cos}[\theta]\right)}, \{\epsilon, 0, 2\}\right], B_0 R_0\right]$

Out[22]= $\text{Vector}\left[\left\{0, \frac{r^2 B_0\, \epsilon}{q[r] R_0} - \frac{r^3\, \text{Cos}[\theta] B_0\, \epsilon^2}{q[r] R_0^2} + O[\epsilon]^3, B_0 R_0\right\}, \left\{0, \frac{B_0\, \epsilon}{q[r] R_0} - \frac{r\, \text{Cos}[\theta] B_0\, \epsilon^2}{q[r] R_0^2} + O[\epsilon]^3, \frac{B_0}{R_0} - \frac{2 (r\, \text{Cos}[\theta] B_0)\, \epsilon}{R_0^2} + \frac{3 r^2\, \text{Cos}[\theta]^2\, B_0\, \epsilon^2}{R_0^3} + O[\epsilon]^3\right\}\right]$

In[23]:= J = Simplify[$\nabla \times B[r,\theta,\varphi]$]

Out[23]= $\text{Vector}\left[\left\{O[\epsilon]^3, O[\epsilon]^3, \frac{B_0\, (2 q[r] - r q'[r])\, \epsilon}{q[r]^2} - \frac{r\, \text{Cos}[\theta] B_0\, \epsilon^2}{q[r] R_0} + O[\epsilon]^3\right\}, \left\{O[\epsilon]^3, O[\epsilon]^3, \frac{B_0\, (2 q[r] - r q'[r])\, \epsilon}{q[r]^2\, R_0^2} + \frac{r\, \text{Cos}[\theta] B_0\, (-5 q[r] + 2 r q'[r])\, \epsilon^2}{q[r]^2\, R_0^3} + O[\epsilon]^3\right\}\right]$

In[24]:= Simplify[$|B[r,\theta,\varphi]|$]

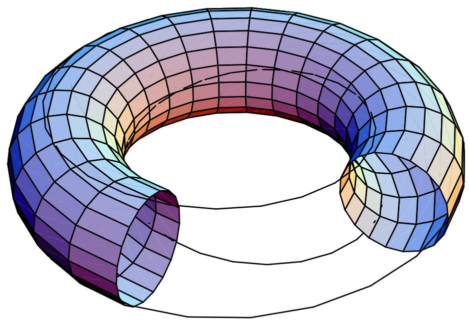
Out[24]= $B_0 - \frac{r\, \text{Cos}[\theta] B_0\, \epsilon}{R_0} + \frac{r^2 \left(1 + 2\, \text{Cos}[\theta]^2\, q[r]^2\right) B_0\, \epsilon^2}{2\, q[r]^2\, R_0^2} + O[\epsilon]^3$

In[25]:= **b = Simplify[B[r, $\hat{\theta}$, φ]]**

Out[25]= $\text{Vector}\left[\left\{O[\epsilon]^3, \frac{r^2\,\epsilon}{q[r]\,R_0} + O[\epsilon]^3, R_0 + r\,\text{Cos}[\theta]\,\epsilon - \frac{r^2\,\epsilon^2}{2\left(q[r]^2\,R_0\right)} + O[\epsilon]^3\right\}, \left\{O[\epsilon]^3, \frac{\epsilon}{q[r]\,R_0} + O[\epsilon]^3, \frac{1}{R_0} - \frac{r\,\text{Cos}[\theta]\,\epsilon}{R_0^2} + \frac{r^2\left(-1 + 2\,\text{Cos}[\theta]^2\,q[r]^2\right)\epsilon^2}{2\,q[r]^2\,R_0^3} + O[\epsilon]^3\right\}\right]$

In[26]:= **Simplify[J × B[r, θ, φ]]**

Out[26]= $\text{Vector}\left[\left\{\frac{r\,B_0^2\,(-2\,q[r] + r\,q'[r])\,\epsilon^2}{q[r]^3\,R_0^2} + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\right\}, \left\{\frac{r\,B_0^2\,(-2\,q[r] + r\,q'[r])\,\epsilon^2}{q[r]^3\,R_0^2} + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\right\}\right]$



In[27]:= **Needs["Utilities‘Notation‘"]**
Symbolize[T]

In[29]:= $\text{Vg} = \frac{c\ T_\perp}{e\,(|\,B[r, \theta, \varphi]\,|)^2}\ b \times \nabla\,(|B[r, \theta, \varphi]\,|);\ \text{Vc} = \frac{c\,2\ T_\parallel}{e\,(|B[r, \theta, \varphi]\,|)}\ b \times ((b \cdot \nabla)\,b);\ \ \text{Vd} = \text{FullSimplify}[\text{Vg} + \text{Vc}];\ \ \text{Vd}[[1]]$

Out[29]= $\left\{-\frac{c\,(2\,T_\parallel + T_\perp)\,\text{Sin}[\theta]\,\epsilon}{e\,B_0\,R_0} + O[\epsilon]^3, -\frac{c\,r\,(2\,T_\parallel + T_\perp)\,\text{Cos}[\theta]\,\epsilon}{e\,B_0\,R_0} + \frac{c\,r^2\,((-2\,T_\parallel + T_\perp)\,q[r] - r\,T_\perp\,q'[r])\,\epsilon^2}{e\,q[r]^3\,B_0\,R_0^2} + O[\epsilon]^3, \frac{c\,r\,(2\,T_\parallel + T_\perp)\,\text{Cos}[\theta]\,\epsilon^2}{e\,q[r]\,B_0\,R_0} + O[\epsilon]^3\right\}$