

On the Perimeter of an Ellipse

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Computing accurate approximations to the perimeter of an ellipse is a favourite problem of mathematicians, attracting luminaries such as Ramanujan [1, 2, 3]. As is well known, the perimeter, \mathcal{P} , of an ellipse with semimajor axis a and semiminor axis b can be expressed exactly as a complete elliptic integral of the second kind, which can also be written as a Gaussian hypergeometric function,

$$\mathcal{P} = 4 a E\left(1 - \frac{b^2}{a^2}\right) = 2 \pi a {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{b^2}{a^2}\right). \quad (1)$$

What is less well known is that the various exact forms attributed to Maclaurin, Gauss-Kummer, and Euler, are related via quadratic hypergeometric transformations [4, 5]. These transformations lead to additional identities, including a particularly elegant formula, symmetric in a and b ,

$$\mathcal{P} = 2 \pi \sqrt{ab} P_{\frac{1}{2}}\left(\frac{a^2 + b^2}{2ab}\right), \quad (2)$$

where $P_\nu(z)$ is a Legendre function.

Approximate formulas can, of course, be obtained by truncating the series representations of exact formulas. For example, Kepler used the geometric mean, $\mathcal{P} \approx 2 \pi \sqrt{ab}$, as a lower bound for the perimeter [6]. In this paper, we examine the properties of a number of approximate formulas, using series methods, polynomial interpolation, rational polynomial approximants, and minimax methods.

■ Cartesian Equation

The Cartesian equation for an ellipse with centre at $(0, 0)$, semimajor axis a , and semiminor axis b reads

$$\text{In[1]:= } \mathcal{E}(x_, y_) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

Introducing the parameter φ into the Cartesian coordinates, as $(x = a \sin(\varphi), y = b \cos(\varphi))$, one verifies that the ellipse equation is satisfied.

$$\text{In[2]:= } \text{Simplify}[\mathcal{E}(a \sin(\varphi), b \cos(\varphi))]$$

$$\text{Out[2]:= } \text{True}$$

■ Arclength

In general, the parametric arclength is defined by

$$\mathcal{L} = \int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} d\varphi \quad (3)$$

The arclength of an ellipse as a function of the parameter φ is an (incomplete) elliptic integral of the second kind.

$$\text{In[3]:= } \mathcal{L}(\varphi_)= \text{With}[\{x = a \sin(\varphi), y = b \cos(\varphi)\},$$

$$\text{Simplify}[\int \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} d\varphi, a > b > 0 \wedge 0 < \varphi < \frac{\pi}{2}]$$

$$\text{Out[3]= } a E\left(\varphi \mid 1 - \frac{b^2}{a^2}\right)$$

Since,

$$\text{In[4]:= } \mathcal{L}(0) = 0$$

$$\text{Out[4]= } \text{True}$$

the arclength of the ellipse is

$$\mathcal{L}(\varphi) = a E(\varphi \mid e^2) \quad (4)$$

where the eccentricity, e , is defined by

$$\text{In[5]:= } e(\mathbf{a}_-, \mathbf{b}_-) = \sqrt{1 - \frac{b^2}{a^2}};$$

■ Perimeter

Since the parameter ranges over $0 \leq \varphi \leq \pi/2$ for one quarter of the ellipse, the perimeter of the ellipse is

$$\text{In[6]:= } \mathcal{P}_1(\mathbf{a}_-, \mathbf{b}_-) = 4 \mathcal{L}\left(\frac{\pi}{2}\right)$$

$$\text{Out[6]= } 4 a E\left(1 - \frac{b^2}{a^2}\right)$$

That is $\mathcal{P} = 4 a E(e^2)$ where $E(m)$ is the complete elliptic integral of the second kind.

□ Alternative Expressions for the Perimeter

The above expression for the perimeter of the ellipse is *unsymmetrical* with respect to the parameters a and b . This is “unphysical” in that both parameters, being lengths of the (major and minor) axes, should be on the same footing. We can expect that a *symmetric* formula, when truncated, will more accurately approximate the perimeter for both $a \geq b$ and $a \leq b$.

Noting that the complete elliptic integral is a gaussian hypergeometric function,

$$\text{In[7]:= } {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; z\right)$$

$$\text{Out[7]= } \frac{2 E(z)}{\pi}$$

one obtains Maclaurin's 1742 formula (see [2])

$$\text{In[8]:= } \mathcal{P}_1(a, b) = 2\pi a {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; e(a, b)^2\right)$$

Out[8]= True

Equivalent alternative expressions for the perimeter of the ellipse can be obtained from quadratic transformation formulas for gaussian hypergeometric functions. For example, using functions.wolfram.com/07.23.17.0106.01,

$$\text{In[9]:= } \text{Simplify}\left[{}_2F_1(\alpha, \beta; 2\beta; z) = \frac{{}_2F_1\left(\alpha, \alpha - \beta + \frac{1}{2}; \beta + \frac{1}{2}; \left(\frac{1 - \sqrt{1-z}}{\sqrt{1-z} + 1}\right)^2\right)}{\left(\frac{1}{2}(\sqrt{1-z} + 1)\right)^{2\alpha}}\right] /.$$

$$\left\{\beta \rightarrow \frac{1}{2}, \alpha \rightarrow -\frac{1}{2}, z \rightarrow e(a, b)^2\right\}, a > b > 0]$$

$$\text{Out[9]= } 4a E\left(1 - \frac{b^2}{a^2}\right) = (a+b)\pi {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(a-b)^2}{(a+b)^2}\right)$$

and noting that

$$\text{In[10]:= } \frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4ab}{(a+b)^2} // \text{Simplify}$$

Out[10]= True

one obtains the following symmetric formula

$$\text{In[11]:= } \mathcal{P}_2(a, b) = \pi(a+b) {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{4ab}{(a+b)^2}\right);$$

first obtained by Ivory (1796), but known as the Gauss-Kummer series (see [2]).

Introducing the homogenous symmetric parameter $h = \frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4ab}{(a+b)^2}$, one has (c.f. mathworld.wolfram.com/Ellipse.html),

$$\text{In[12]:= } \pi(a+b) {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; h\right) // \text{FunctionExpand} // \text{Simplify}$$

$$\text{Out[12]= } 2(a+b)(2E(h) + (h-1)K(h))$$

Explicitly, the Gauss-Kummer series reads

$$\text{In[13]:= } \mathcal{P}_3(a, b) = \text{FullSimplify}[\mathcal{P}_2(a, b) // \text{FunctionExpand}, a > b > 0]$$

$$\text{Out[13]= } 4(a+b)E\left(1 - \frac{4ab}{(a+b)^2}\right) - \frac{8abK\left(1 - \frac{4ab}{(a+b)^2}\right)}{a+b}$$

Instead, using functions.wolfram.com/07.23.17.0103.01, one obtains Euler's 1773 formula (see also [2]):

$$\text{In[14]:= } {}_2F_1(\alpha, \beta; 2\beta; z) = \frac{{}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \beta + \frac{1}{2}; \frac{z^2}{(2-z)^2}\right)}{\left(1 - \frac{z}{2}\right)^\alpha} /.$$

$$\left\{\beta \rightarrow \frac{1}{2}, \alpha \rightarrow -\frac{1}{2}, z \rightarrow e(a, b)^2\right\} // \text{Simplify}$$

$$\text{Out[14]= } 4E\left(1 - \frac{b^2}{a^2}\right) = \sqrt{\frac{2b^2}{a^2}} + 2\pi {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}; 1; \frac{(a^2 - b^2)^2}{(a^2 + b^2)^2}\right)$$

The hidden symmetry with respect to the interchange $a \leftrightarrow b$ is revealed.

$$\text{In[15]:= } \text{FullSimplify}[\%, b > a > 0]$$

$$\text{Out[15]= } bE\left(1 - \frac{a^2}{b^2}\right) = aE\left(1 - \frac{b^2}{a^2}\right)$$

Defining

$$\text{In}[16]:= \mathcal{P}_4(\mathbf{a}_-, \mathbf{b}_-) = \pi \sqrt{2(a^2 + b^2)} {}_2F_1\left(\frac{1}{4}, -\frac{1}{4}; 1; \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2\right);$$

one can directly check the formula.

$$\text{In}[17]:= \text{Simplify}[\mathcal{P}_4(a, b) = \mathcal{P}_1(a, b) // \text{FunctionExpand}, a > b > 0]$$

$$\text{Out}[17]= \text{True}$$

□ Other identities

There are many other possible transformation formulas that can be applied to obtain alternative expressions for the perimeter. For example, using functions.wolfram.com/07.23.17.0054.01 one obtains the following formula,

$$\begin{aligned} \text{In}[18]:= \mathcal{P}_5(\mathbf{a}_-, \mathbf{b}_-) &= \mathcal{P}_2(a, b) /. {}_2F_1(\mathbf{a}_-, \mathbf{b}_-, \mathbf{c}_-, \mathbf{z}_-) \rightarrow (1-z)^{-a-b+c} {}_2F_1(c-a, c-b; c; z) \\ \text{Out}[18]= & \frac{16 a^2 b^2 \pi {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 1; 1 - \frac{4ab}{(a+b)^2}\right)}{(a+b)^3} \end{aligned}$$

The perimeter can also be expressed in terms of Legendre functions (see sections 8.13 and 15.4 of [7]). For example, using 15.4.15 of [7] one obtains an elegant and simple symmetric formula

$$\begin{aligned} \text{In}[19]:= \mathcal{P}_6(\mathbf{a}_-, \mathbf{b}_-) &= \text{Simplify}[\mathcal{P}_2(a, b) /. {}_2F_1(\mathbf{a}_-, \mathbf{b}_-, \mathbf{c}_-, \mathbf{x}_-) \rightarrow \\ & \Gamma(a-b+1) (1-x)^{-b} (-x)^{\frac{b-a}{2}} P_{-b}^{b-a}\left(\frac{1+x}{1-x}\right) /; c = a-b+1, a > 0 \wedge b > 0] \\ \text{Out}[19]= & 2\sqrt{ab} \pi P_{\frac{1}{2}}\left(\frac{a^2+b^2}{2ab}\right) \end{aligned}$$

Alternatively, this result follows directly from 8.13.6 of [7] with $e^\eta = a/b \implies \cosh(\eta) = \frac{a^2+b^2}{2ab}$. This form can be used to prove that the perimeter of an ellipse is a homogenous mean (c.f. [8]), extending the arithmetic-geometric mean (AGM) already used as a tool for computing elliptic integrals [9].

Using functions.wolfram.com/07.07.26.0001.01, this gives yet another formula involving complete elliptic integrals.

$$\begin{aligned} \text{In}[20]:= \mathcal{P}_7(\mathbf{a}_-, \mathbf{b}_-) &= \\ & \mathcal{P}_6(a, b) /. P_{\nu_-}(\mathbf{z}_-) \rightarrow {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) // \text{FunctionExpand} // \text{Simplify} \\ \text{Out}[20]= & 4\sqrt{ab} \left(2E\left(-\frac{(a-b)^2}{4ab}\right) - K\left(-\frac{(a-b)^2}{4ab}\right)\right) \end{aligned}$$

□ Comparisons

Here we compare the seven formulas obtained above for $b = 2a$,

$$\begin{aligned} \text{In}[21]:= \text{Simplify}\{\mathcal{P}_1(a, 2a), \mathcal{P}_2(a, 2a), \mathcal{P}_3(a, 2a), \\ \mathcal{P}_4(a, 2a), \mathcal{P}_5(a, 2a), \mathcal{P}_6(a, 2a), \mathcal{P}_7(a, 2a)\}, a > 0] \\ \text{Out}[21]= \left\{4a E(-3), 3a \pi {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{9}\right), \frac{4}{3}a \left(9E\left(\frac{1}{9}\right) - 4K\left(\frac{1}{9}\right)\right), \right. \\ \left. \sqrt{10} a \pi {}_2F_1\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{9}{25}\right), \frac{64}{27} a \pi {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{9}\right), \right. \\ \left. 2\sqrt{2} a \pi P_{\frac{5}{2}}\left(\frac{5}{4}\right), 4\sqrt{2} a \left(2E\left(-\frac{1}{8}\right) - K\left(-\frac{1}{8}\right)\right)\right\} \end{aligned}$$

```

In[22]:= N[%]
Out[22]:= {9.688448221 a, 9.688448221 a, 9.688448221 a,
          9.688448221 a, 9.688448221 a, 9.688448221 a}

In[23]:= Equal @@ %
Out[23]:= True

and for  $b = a/3$ .

In[24]:= Simplify[{ $\mathcal{P}_1(a, \frac{a}{3}), \mathcal{P}_2(a, \frac{a}{3}), \mathcal{P}_3(a, \frac{a}{3}),$ 
 $\mathcal{P}_4(a, \frac{a}{3}), \mathcal{P}_5(a, \frac{a}{3}), \mathcal{P}_6(a, \frac{a}{3}), \mathcal{P}_7(a, \frac{a}{3})$ },  $a > 0$ ]
Out[24]:= { $4 a E(\frac{8}{9}), \frac{4}{3} a \pi {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{4}),$ 
 $\frac{2}{3} a (8 E(\frac{1}{4}) - 3 K(\frac{1}{4})), \frac{2}{3} \sqrt{5} a \pi {}_2F_1(\frac{1}{4}, -\frac{1}{4}; 1; \frac{16}{25}),$ 
 $\frac{3}{4} a \pi {}_2F_1(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{4}), \frac{2 a \pi P_{\frac{1}{2}}(\frac{5}{3})}{\sqrt{3}}, \frac{a (8 E(-\frac{1}{3}) - 4 K(-\frac{1}{3}))}{\sqrt{3}}$ }

In[25]:= N[%]
Out[25]:= {4.454964407 a, 4.454964407 a, 4.454964407 a,
          4.454964407 a, 4.454964407 a, 4.454964407 a}

In[26]:= Equal @@ %
Out[26]:= True

```

■ Numerical Approximation

At www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html [1] one is encouraged to search for “an efficient formula using only the four algebraic operations (if possible, avoiding even square-root) with a maximum error below 10 parts per million. It would be also nice if such a formula were exact for both the circle and the degenerate flat ellipse.”

The Gauss-Kummer series expressed as a function of the homogenous variable $h = 1 - 4 a b / (a + b)^2$, reads

```

In[27]:= GaussKummer[h_] =  $\frac{\mathcal{P}_2(a, b)}{a + b} /. (a + b) \rightarrow 2 \sqrt{a b} / \sqrt{1 - h}$ 
Out[27]:=  $\pi {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; h)$ 

```

□ Series expansions

The series expansion about $h = 0$ is useful for small h .

```

In[28]:= GaussKummer[h] + O[h]^9
Out[28]:=  $\pi + \frac{\pi h}{4} + \frac{\pi h^2}{64} + \frac{\pi h^3}{256} + \frac{25 \pi h^4}{16384} +$ 
 $\frac{49 \pi h^5}{65536} + \frac{441 \pi h^6}{1048576} + \frac{1089 \pi h^7}{4194304} + \frac{184041 \pi h^8}{1073741824} + O(h^9)$ 

```

Around $h = 1$, terms in $\log(1 - h)$ arise.

```

In[29]:= Simplify[Series[GaussKummer[h], {h, 1, 2}], 0 < h < 1]
Out[29]:=  $4 + (h - 1) + \frac{1}{16} (-2 \log(1 - h) - 4 \psi^{(0)}(\frac{3}{2}) - 4 \gamma + 3)(h - 1)^2 + O((h - 1)^3)$ 

```

Using functions.wolfram.com/07.23.06.0015.01 we can obtain the general term of this series (c.f. 17.3.33-17.3.36 of [7]),

$$\begin{aligned}
 \text{In[30]:= } & \text{GaussKummer}[h] /. _2F_1(a_, b_ ; c_ ; z_) := \text{With}[\{n = c - a - b\}, \\
 & \frac{\Gamma(a + b + n)}{\Gamma(a) \Gamma(b)} \left(\sum_{k=0}^{\infty} \frac{(a + n)_k (b + n)_k}{k! (k + n)!} (-\log(1 - z) + \psi(k + 1) + \right. \\
 & \quad \left. \psi(k + n + 1) - \psi(a + k + n) - \psi(b + k + n)) (1 - z)^k \right) (z - 1)^n + \\
 & \frac{(n - 1)! \Gamma(a + b + n)}{\Gamma(a + n) \Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k (1 - z)^k}{k! (1 - n)_k}] // \text{Simplify} \\
 \text{Out[30]:= } & \frac{1}{4} \left(\left(\sum_{k=0}^{\infty} \frac{(1 - h)^k \left(\frac{3}{2}\right)_k^2 (-\log(1 - h) + \psi^{(0)}(k + 1) - 2\psi^{(0)}\left(k + \frac{3}{2}\right) + \psi^{(0)}(k + 3))}{k! (k + 2)!} \right) \right. \\
 & \quad \left. (h - 1)^2 + 4(h + 3) \right)
 \end{aligned}$$

□ Polynomial Approximants

■ Linear Approximant

From the exact values at $h = 0$,

$$\text{In[31]:= GaussKummer}[0]$$

$$\text{Out[31]:= } \pi$$

and at $h = 1$,

$$\text{In[32]:= GaussKummer}[1]$$

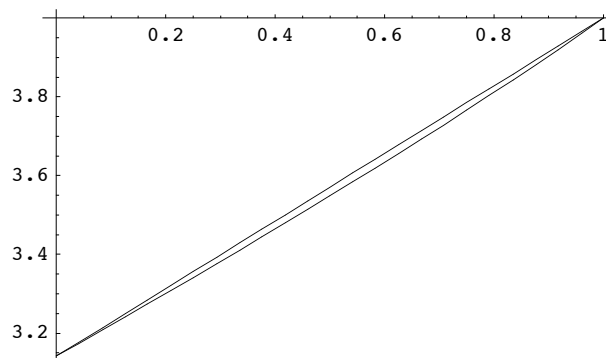
$$\text{Out[32]:= } 4$$

one constructs the linear *extreme perfect* approximant.

$$\text{In[33]:= Linear}[h_] = (1 - h) \text{GaussKummer}[0] + h \text{GaussKummer}[1] // \text{Simplify}$$

$$\text{Out[33]:= } \pi - h(-4 + \pi)$$

$$\text{In[34]:= Plot}[\{\text{GaussKummer}[h], \text{Linear}[h]\}, \{h, 0, 1\}]$$



■ Quadratic Approximant

The quadratic approximant, exact at $h = 0, 1/2, 1$,

In[35]:= **Table**[{*h*, **GaussKummer**[*h*]}, {*h*, 0, 1, $\frac{1}{2}$ }] // **FullSimplify**

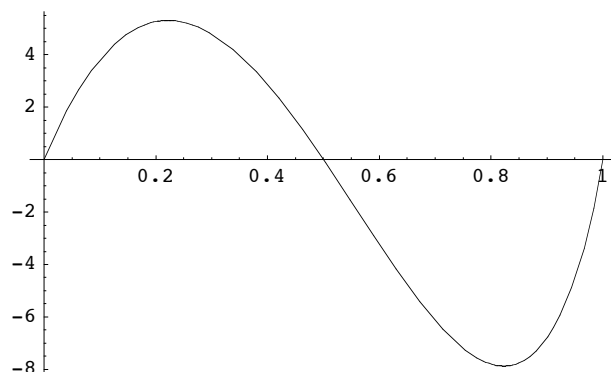
$$\text{Out[35]} = \begin{pmatrix} 0 & \pi \\ \frac{1}{2} & \frac{\sqrt{\frac{\pi}{2}} \Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} + \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{5}{4})}{\sqrt{\pi}} \\ 1 & 4 \end{pmatrix}$$

In[36]:= **Quadratic**[*h_*] = **InterpolatingPolynomial**[% , *h*] // *N*

Out[36]= (0.08918191962 (*h* - 0.5) + 0.8138163866) *h* + 3.141592654

has a maximum absolute relative error of $\leq 8 \times 10^{-4}$.

In[37]:= **Plot**[$10^4 \left(1 - \frac{\text{Quadratic}[h]}{\text{GaussKummer}[h]} \right)$, {*h*, 0, 1}]



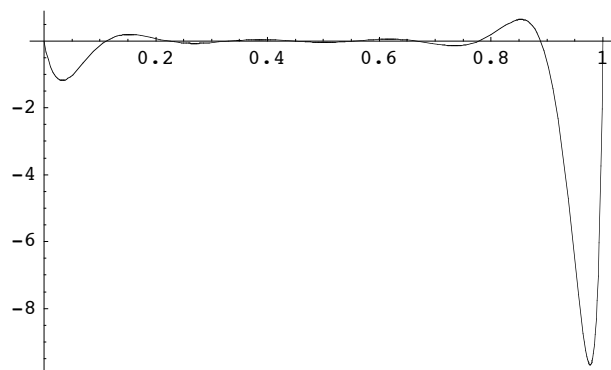
■ ***n*th-order polynomial Approximant**

Here is the *n*th-order “even-tempered” polynomial approximant, exact at *h* = *m* / *n* for *m* = 0, 1, ..., *n*.

In[38]:= **poly**[*n_*] := **poly**[*n*] = **Function**[*h*, **Evaluate** @ **InterpolatingPolynomial**[
 $N @ \text{Table}[\{h, \text{GaussKummer}[h]\}, \{h, 0, 1, \frac{1}{n}\}], h]$

The 9th-order approximant has a maximum absolute relative error of $< 10 \times 10^{-6}$.

In[39]:= **Plot**[$10^6 \left(1 - \frac{\text{poly}[9][h]}{\text{GaussKummer}[h]} \right)$, {*h*, 0, 1}, **PlotRange** → **All**, **PlotPoints** → 30]



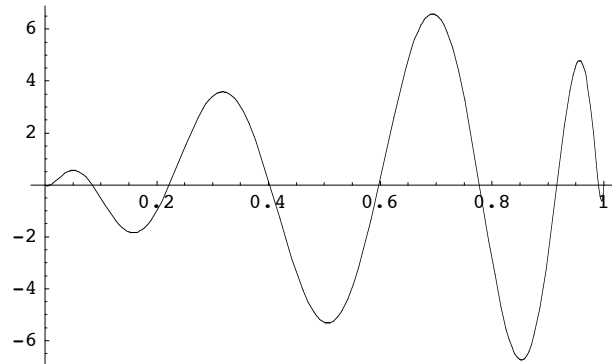
■ **Chebyshev polynomial Approximant**

Sampling the Gauss-Kummer function at the zeros of $T_n(2x - 1)$, which are at $x_m = \cos^2((m + 1/4) \frac{\pi}{n})$, yields a Chebyshev polynomial approximant.

```
In[40]:= Chebyshevpoly[n_] :=
  Chebyshevpoly[n] = Function[h, Evaluate@InterpolatingPolynomial[
    N@Join[{{0, GaussKummer[0]}, {1, GaussKummer[1]}},
    Table[{cos^2((m + 1/4) π/n), GaussKummer[cos^2((m + 1/4) π/n)]},
    {m, n}]], h]]
```

The 8th-order approximant has a maximum absolute relative error of $\lesssim 7 \times 10^{-6}$.

```
In[41]:= Plot[10^6 (1 - Chebyshevpoly[8][h] / GaussKummer[h]), {h, 0, 1}, PlotRange -> All]
```



□ Rational Approximation

After loading the package (stub),

```
In[42]:= << NumericalMath`
```

one obtains a family of $[N, M]$ rational polynomial minimax approximations.

```
In[43]:= GKapprox[n_, m_] := GKapprox[n, m] = Function[h, Evaluate[
  MiniMaxApproximation[GaussKummer[h], {h, {0, 1}, n, m}][[2, 1]]]
```

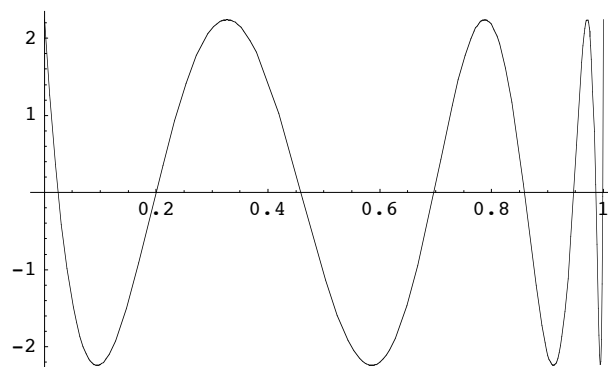
For example, the $[4,3]$ minimax approximation,

```
In[44]:= GKapprox[4, 3][h]
```

```
Out[44]:= (-0.08111828563 h^4 + 0.273498199 h^3 +
  1.771628564 h^2 - 5.055401264 h + 3.14159195) /
  (-0.1414596605 h^3 + 1.013205136 h^2 - 1.859195682 h + 1)
```

has (absolute) relative error $\lesssim 2.3 \times 10^{-7}$, but is not “extreme perfect”.

```
In[45]:= Plot[10^7 (1 - GKapprox[4, 3][h] / GaussKummer[h]), {h, 0, 1}]
```



Using the linear approximant, $4h + \pi(1 - h)$, and noting that $h(1 - h)$ vanishes at both $h = 0$ and $h = 1$, leads to an optimal $[N + 2, M]$ extreme perfect approximant of the form

$$\pi {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; h\right) \approx 4h + \pi(1 - h) + \alpha h(1 - h) \frac{\prod_{i=1}^N (h - p_i)}{\prod_{j=1}^M (h - q_j)},$$

where the parameters α , $\{p_i\}_{i=1,\dots,N}$, and $\{q_j\}_{j=1,\dots,M}$ need to be determined. Implementation of the approximant is immediate.

```
In[46]:= EllipseApproximant[alpha_, p_List, q_List] :=
Function[h, Evaluate[4 h + pi (1 - h) + alpha h (1 - h)
Times @@ (h - p)
Times @@ (h - q)]]
```

After uniformly sampling the Gauss-Kummer function,

```
In[47]:= {xdata, ydata} = Table[{h, GaussKummer[h]}, {h, 0, 1, 0.001}] // Transpose;
```

one can use NMinimize and the ∞ -norm to obtain the accurate approximants. For example, the (almost) optimal $[3, 2]$ approximant is computed using

```
In[48]:= NMinimize[
||ydata - EllipseApproximant[alpha, {p}, {q, r}][xdata]||_infinity,
{alpha, 0.22, 0.24,
p, 1.25, 1.35,
q, 3.4, 3.5,
r, 1.15, 1.25}]
```

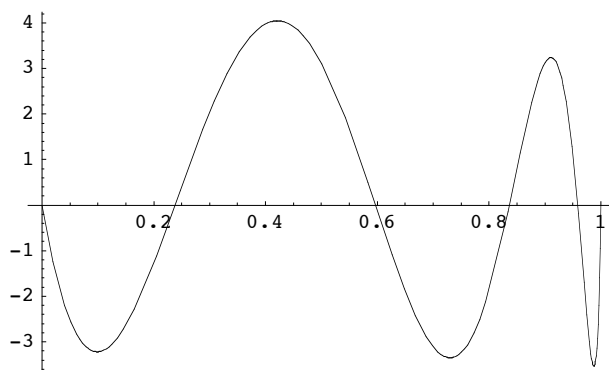
```
Out[48]= {0.0000140975141,
{p -> 1.285457885, q -> 3.475000451, r -> 1.196711294, alpha -> 0.2354557322}}
```

leading to

```
In[49]:= EllipseApproximant[alpha, {p}, {q, r}][h] /. Last[%]
Out[49]= 0.2354557322 (h - 1.285457885) h (1 - h)
(h - 3.475000451) (h - 1.196711294) + pi (1 - h) + 4 h
```

This simple approximant has (absolute) relative error $\lesssim 4 \times 10^{-6}$.

```
In[50]:= Plot[10^6 (1 - % / GaussKummer[h]), {h, 0, 1}]
```



■ Conclusions

Computing the perimeter of an ellipse using a simple set of approximants demonstrates that *Mathematica* is an ideal tool for developing accurate approximants to special function. In particular,

- all special functions of mathematical physics are built-in and can be evaluated to arbitrary precision for general complex parameters and variables;
- standard analytical methods—such as symbolic integration, summation, series and asymptotic expansions, and polynomial interpolation—are available;
- properties of special functions—such as identities and transformations—are available at MathWorld [9] and the Wolfram functions Site [10] and, because these properties are expressed in *Mathematica* syntax, can be used directly;
- relevant built-in numerical methods include rational polynomial approximants, minimax methods, and numerical optimization for arbitrary norms;
- visualization of approximants can be used to estimate the quality of approximants; and
- combining these approaches is straightforward and leads, in a natural way, to optimal approximants.

■ References

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