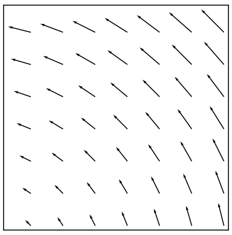
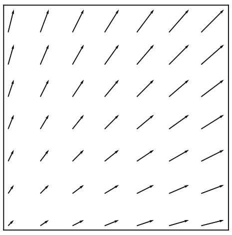
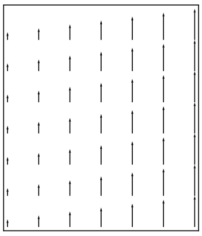
ElectroMagnetism — Assignment 2

Solutions are to be handed in by 5pm, Monday September 14

1. Consider the following three vector fields:


(1)


(2)


(3)

(a) Which of these is *solenoidal*, *i.e.*, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$? [2]

Answer: From the figure, the three fields are easily determined. Computing the divergences,

$$\nabla \cdot \{-y, x, 0\}$$

$$0$$

$$\nabla \cdot \{x, y, 0\}$$

$$2$$

$$\nabla \cdot \{0, x, 0\}$$

$$0$$

we see that (1) and (3) are solenoidal.

(b) Which of these is *irrotational*, *i.e.*, $\boldsymbol{\nabla} \times \boldsymbol{v} = \mathbf{0}$? [2]

Answer: Computing the curls,

$$\nabla \times \{-y, x, 0\}$$

$$\{0, 0, 2\}$$

$$\nabla \times \{x, y, 0\}$$

$$\{0, 0, 0\}$$

$$\nabla \times \{0, x, 0\}$$

$$\{0, 0, 1\}$$

Hence only (2) is irrotational.

2. A hollow spherical shell of inner radius a and outer radius b has charge density $\rho = k/r^2$ in the region $a \le r \le b$. Compute and display the (magnitude of the) electric field and the potential for $r \ge 0$. Check that $\nabla \cdot \boldsymbol{E} = \rho / \epsilon_0$. [4]

Answer: The charge enclosed in a Gaussian surface of radius r for $a \le r \le b$ is

$$\boldsymbol{\rho(r) = \frac{k}{r^2};}$$

$$\boldsymbol{q(\text{r\_}) = 4\pi \int_a^r \rho(r)\, r^2\, dr}$$

$$4\pi k (r - a)$$

From the spherical symmetry of the problem, $\boldsymbol{E} = E_r \hat{\boldsymbol{r}}$. Using Gauss's Law, the electric field for $0 \le r \le a$ is zero (no charge is enclosed for $0 \le r \le a$), the radial component of the electric field for $a < r \le b$ is

$$\textbf{Solve}\left[\mathcal{E}\, 4\pi r^2 == \frac{q(r)}{\epsilon_0}, \mathcal{E}\right] \textbf{ // First // Expand}$$

$$\left\{\mathcal{E} \to \frac{k}{r\,\epsilon_0} - \frac{a\,k}{r^2\,\epsilon_0}\right\}$$

and the radial component of the electric field for $r > b$ is

$$\textbf{Solve}\left[\mathcal{E}\, 4\pi r^2 == \frac{q(b)}{\epsilon_0}, \mathcal{E}\right] \textbf{ // First // Expand}$$

$$\left\{\mathcal{E} \to \frac{b\,k}{r^2\,\epsilon_0} - \frac{a\,k}{r^2\,\epsilon_0}\right\}$$

Compute the potential, starting at ∞, using $V(r) = -\int_\infty^r \boldsymbol{E} \cdot d\boldsymbol{r} = \int_r^\infty E_r\, dr$. For $r > b > 0$,

$$\textbf{Assuming}\left[r > b > 0, -\int_\infty^r \left(\frac{b\,k}{r^2\,\epsilon_0} - \frac{a\,k}{r^2\,\epsilon_0}\right) dr\right]$$

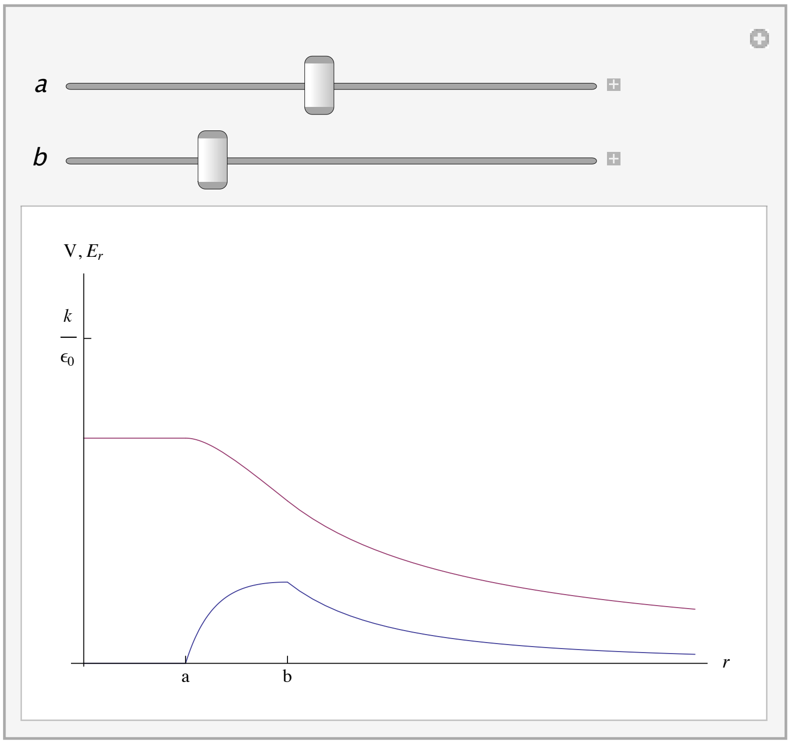
$$-\frac{k\,(a - b)}{r\,\epsilon_0}$$

and for $b > r > a > 0$,

$$\textbf{Assuming}\left[b > r > a > 0, -\frac{k\,(a - b)}{b\,\epsilon_0} - \int_b^r \left(\frac{k}{r\,\epsilon_0} - \frac{a\,k}{r^2\,\epsilon_0}\right) dr\right] \textbf{ // Expand}$$

$$-\frac{a\,k}{r\,\epsilon_0} - \frac{k \log\left(\frac{r}{b}\right)}{\epsilon_0} + \frac{k}{\epsilon_0}$$

Here is a plot of the potential and radial electric field as a function of r.



As a check, for the potential,

$$V(\mathrm{r\_}) := \frac{k}{\epsilon_0}\begin{cases}\log\left(\frac{b}{a}\right) & 0 < r \le a\\ 1+\log\left(\frac{b}{r}\right)-\frac{a}{r} & a < r \le b\\ \frac{b-a}{r} & r > b\end{cases}$$

the radial electric field, $E_r = -\partial_r V(r)$, is

$$\mathcal{E}(\mathrm{r\_}) = \mathrm{Simplify}[-\partial_r V(r), r > 0]$$

$$-\frac{k\left(\begin{cases}0 & a \ge r\\ \frac{a-r}{r^2} & a < r \bigwedge b \ge r\\ \frac{a-b}{r^2} & \text{True}\end{cases}\right)}{\epsilon_0}$$

Check that $\nabla \cdot \boldsymbol{E} = r^{-2}\,\partial_r\left(r^2 E_r\right) = \rho/\epsilon_0$,

$$\mathrm{Simplify}\left[\frac{1}{r^2}\,\partial_r\left(r^2\,\mathcal{E}(r)\right) = \frac{\rho(r)}{\epsilon_0}, a < r \le b\right]$$

True

$$\mathrm{Simplify}\left[\frac{1}{r^2}\,\partial_r\left(r^2\,\mathcal{E}(r)\right) = 0, r < a\right]$$

True

$$\mathbf{Simplify}\left[\frac{1}{r^2}\,\partial_r\left(r^2\,\mathcal{E}(r)\right) = 0,\, r > b\right]$$

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True
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3. Consider a thin evacuated spherical shell of radius R which has surface potential

$$V(R,\,\theta) = k\sin^2\left(\frac{\theta}{2}\right) \tag{1}$$

where k is a constant and $0 \le \theta \le \pi$ is the angle in spherical polar coordinates measured from the z-axis. In the case of azimuthal symmetry, the *general solution* to Laplace's equation $\nabla^2 V = 0$ is

$$V(r,\,\theta) = \sum_{l=0}^{\infty}\left(A_l\,r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta)) \tag{2}$$

where $P_l(x)$ are the Legendre polynomials: $P_0(x) = 1, P_1(x) = x, \ldots$

Use equations (1) and (2) and the *orthogonality* of the Legendre polynomials to determine the *internal* potential, *i.e.*, $V(r,\,\theta)$ for $r \le R$. Hint: $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\,(1 - \cos(\theta)) = \frac{1}{2}\,(P_0(\cos(\theta)) - P_1(\cos(\theta)))$. [6]

Answer: As $r \to 0$, $A_l\,r^l + B_l\,r^{-l-1} \longrightarrow B_l\,r^{-l-1} \longrightarrow \infty$. $V(r,\,\theta)$ finite $\Rightarrow B_l = 0$ for $l \ge 0$.

Equating (1) and (2) on the surface of the sphere determines all A_l. Setting $r = R$ in (2) we have

$$V(R,\,\theta) = \sum_{l=0}^{\infty} A_l\,R^l\,P_l(\cos(\theta)) = A_0\,P_0(\cos(\theta)) + A_1\,R\,P_1(\cos(\theta)) + \cdots = k\sin^2\left(\frac{\theta}{2}\right) = \frac{k}{2}\,(P_0(\cos(\theta)) - P_1(\cos(\theta)))$$

Using the orthogonality of the Legendre polynomials, $A_0 = k/2$, $A_1 = -k/(2\,R)$ and $A_{l\ge 2} = 0$. Hence, for $0 \le r \le R$,

$$V(r,\,\theta) = \frac{k}{2}\,P_0(\cos(\theta)) - \frac{k\,r}{2\,R}\,P_1(\cos(\theta)) = \frac{k}{2}\left(1 - \frac{r}{R}\cos(\theta)\right)$$

4. For a point charge:

(a) Calculate $\boldsymbol{\nabla}\times\boldsymbol{E}$ directly. [1]

Answer: Writing

$$\mathbb{E} = \frac{q}{4\,\pi\,\epsilon_0}\,\frac{1}{\left(x^2 + y^2 + z^2\right)^{3/2}}\,\{x,\, y,\, z\};$$

the curl in cartesian coordinates is

$$\nabla\times\mathbb{E}$$

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{0, 0, 0}
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Alternatively, since $\boldsymbol{E} \equiv E_r\,\hat{\boldsymbol{r}} + E_\theta\,\hat{\boldsymbol{\theta}} + E_\phi\,\hat{\boldsymbol{\phi}}$, we have

$$\left\{\mathbb{E}_r = \frac{q}{4\,\pi\,\epsilon_0\,r^2},\, \mathbb{E}_\theta = 0,\, \mathbb{E}_\phi = 0\right\};$$

Computing the curl in spherical coordinates we obtain the same answer.

$$\frac{1}{r\sin(\theta)}\left(\frac{\partial\left(\sin(\theta)\,\mathbb{E}_\phi\right)}{\partial\theta}-\frac{\partial\mathbb{E}_\theta}{\partial\phi}\right)e_r+$$

$$\frac{1}{r\sin(\theta)}\left(\frac{\partial\mathbb{E}_r}{\partial\phi}-\sin(\theta)\frac{\partial\left(r\,\mathbb{E}_\phi\right)}{\partial r}\right)e_\theta+\frac{1}{r}\left(\frac{\partial(r\,\mathbb{E}_\theta)}{\partial r}-\frac{\partial\mathbb{E}_r}{\partial\theta}\right)e_\phi$$

0

(b) Compute $\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l}$, in spherical coordinates using $d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin(\theta)\,d\phi\,\hat{\boldsymbol{\phi}}$. [1]

Answer: Because the field is spherically symmetric, $\boldsymbol{E}\cdot d\boldsymbol{l} \equiv E_r\,\hat{\boldsymbol{r}}\cdot d\boldsymbol{l} = E_r\,dr$, and the line integral only depends on the radii, r_a and r_b:

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{r_a}^{r_b} E_r\,dr = \frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right).$$

(c) Show that the integral in (b) around *any* closed path is zero. [1]

Answer: For a closed path, $r_a = r_b$ so the integral vanishes.

(d) Compute $\boldsymbol{\nabla}\times\boldsymbol{E}$ from (c) by applying Stokes' theorem. [1]

Answer: Stokes' theorem tells us that

$$\oint_{C=\partial S}\boldsymbol{E}\cdot d\boldsymbol{l} = 0 = \int_S \boldsymbol{\nabla}\times\boldsymbol{E}\cdot d\boldsymbol{S}$$

for *any* surface S. Hence we conclude that $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$.

(e) Show how you can conclude that $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$, for *any* static charge distribution. [2]

Answer: We have shown that $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$ for a point charge centred at the origin.

However this result makes no reference to what is, after all, a perfectly *arbitrary* choice of coordinates; they hold no matter *where* the charge is located. To see this explicitly, the potential of a charge located at $\boldsymbol{a}$ measured at $\boldsymbol{r}$ is

$$V_{\boldsymbol{a}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{a}|}$$

and the corresponding electric field is

$$\boldsymbol{E}_{\boldsymbol{a}}(\boldsymbol{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{a}|}\right) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3}$$

which can be simply obtained by the (linear) translation of the origin $\boldsymbol{r}\to\boldsymbol{r}-\boldsymbol{a}$. Using *Mathematica* we verify this explicitly

$$\mathbb{E} = \nabla\left(\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\right)$$

$$\left\{-\frac{x-a}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}},\right.$$

$$\left.-\frac{y-b}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}},-\frac{z-c}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}}\right\}$$

So $\boldsymbol{\nabla}\times\boldsymbol{E}_{\boldsymbol{a}}(\boldsymbol{r})$ is

$\nabla \times \mathbb{E}$

$\{0, 0, 0\}$

This can be seen more directly using the chain rule by noting that derivatives such as ∂_x are unaffected by linear translation of coordinates:

$$\frac{\partial f(x-a)}{\partial x} \equiv \left.\frac{\partial f(x)}{\partial x}\right|_{x \to x-a}$$

or, in *Mathematica* notation,

$$\frac{\partial f(x-a)}{\partial x} == \left(\frac{\partial f(x)}{\partial x} /. x \to x-a\right)$$

True

Moreover, if we have many charges, the principle of *superposition* states that the total field is the (vector) sum of their individual fields, $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots$, so

$$\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots) = \nabla \times \boldsymbol{E}_1 + \nabla \times \boldsymbol{E}_2 + \ldots = \mathbf{0}$$

Consequently, $\nabla \times \boldsymbol{E} = \mathbf{0}$ (and $\oint_{C=\partial S} \boldsymbol{E} \cdot d\boldsymbol{l} = 0$) holds for *any* static charge distribution.

5. Two vector identities.

(a) Show that $\nabla \times (\nabla \psi) = \mathbf{0}$, using Stokes' theorem.[2]

Answer: Using Stokes' theorem

$$\int_S (\nabla \times (\nabla \psi)) \cdot d\boldsymbol{S} = \oint_{\partial S} \nabla \psi \cdot d\boldsymbol{l} = \oint_{\partial S} d\psi = 0$$

since $d\psi = \nabla \psi \cdot d\boldsymbol{l}$. Since the surface S is *arbitrary* $\nabla \times (\nabla \psi) = \mathbf{0}$.

(b) Show that $\nabla \cdot \nabla \times \boldsymbol{A} = 0$, for $\boldsymbol{A}$ an arbitrary differentiable vector field by applying Stokes' theorem to an arbitrary *closed* surface S and then using Gauss' theorem. [3]

Answer: For an arbitrary *closed* surface Stokes' theorem yields

$$\oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{l} = \oint_S \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = 0.$$

Apply Gauss' theorem to conclude that

$$\oint_{S=\partial V} \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \nabla \times \boldsymbol{A}\, dV = 0.$$

Since the surface S is *arbitrary*, the volume is also arbitrary and hence the integrand of the volume integral must vanish identically, *i.e.*, $\nabla \cdot \nabla \times \boldsymbol{A} = 0$.

6. Maxwell's equations in free space ($\boldsymbol{J} = \mathbf{0}$ and $\rho = 0$) read

$$\begin{array}{cc} \nabla \cdot \boldsymbol{E} = 0 & \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} & \nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \end{array}$$

(a) Using Maxwell's equations derive the wave equation for $\boldsymbol{B}$. [3]

Answer: Taking the curl of both sides of $\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ we obtain

$$\nabla \times \nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \nabla \times \frac{\partial \boldsymbol{E}}{\partial t}$$

Rearrange the left-hand side using the vector field identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}.$$

and the commutativity of $\boldsymbol{\nabla}$ and $\partial / \partial t$ on the right-hand side to obtain

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = \mu_0 \, \epsilon_0 \, \frac{\partial(\boldsymbol{\nabla} \times \boldsymbol{E})}{\partial t}.$$

Using Maxwell's equations for $\boldsymbol{\nabla} \cdot \boldsymbol{B}$ and $\boldsymbol{\nabla} \times \boldsymbol{E}$ we then have

$$\nabla^2 \boldsymbol{B} = \mu_0 \, \epsilon_0 \, \frac{\partial^2 \boldsymbol{B}}{\partial t^2}.$$

Note that $\boldsymbol{E}$ and $\boldsymbol{B}$ both satisfy an *identical* wave equation.

(b) Show that the divergence of the second pair of Maxwell's equations are consistent by using the vector field identity $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = 0$ which holds for *any* vector field $\boldsymbol{A}$. [2]

Answer: Taking the divergence of both sides of $\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ and using the commutativity of $\boldsymbol{\nabla}$ and $\partial / \partial t$, we obtain

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} = \;\; 0 \equiv -\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial(\boldsymbol{\nabla} \cdot \boldsymbol{B})}{\partial t} = 0.$$

Similarly,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{B} = \;\; 0 \equiv \mu_0 \, \epsilon_0 \, \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \, \epsilon_0 \, \frac{\partial(\boldsymbol{\nabla} \cdot \boldsymbol{E})}{\partial t} = 0.$$

Hence both equations are consistent.

7. Consider the plane-wave electric field $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \, e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$, where $\boldsymbol{k} = (k_x, k_y, k_z)$ is the wave-vector, $\boldsymbol{r} = (x, y, z)$, and ω is the angular frequency.

(a) Show that $\boldsymbol{E}(\boldsymbol{r}, t)$ is a solution of the wave equation. Obtain a relation between $k = |\boldsymbol{k}|$, ω, and the phase velocity. [2]

Since $\frac{\partial^2}{\partial t^2} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -\omega^2 \, e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$ and $\frac{\partial^2}{\partial x^2} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -k_x^2 \, e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$ we have that

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}, t) = \frac{k^2}{\omega^2} \, \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t^2}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2 = |\boldsymbol{k}|^2$.

For $\boldsymbol{E}(\boldsymbol{r}, t)$ to satisfy the wave equation we deduce the dispersion relation, $v_\phi = (\pm) \, \frac{\omega}{k}$.

(b) Use $\boldsymbol{E}(\boldsymbol{r}, t)$, (plus an analogous expression for $\boldsymbol{B}$), to convert Maxwell's equations from vector differential equations to plain vector equations. What is the physical meaning of these vector relations? [3]

$\partial_x e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = i \, k_x \, e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} \Rightarrow \;\; \boldsymbol{\nabla} \leftrightarrow i \, \boldsymbol{k}$ and $\partial_t e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -i \, \omega \, e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} \Rightarrow \;\; \partial_t \leftrightarrow -i \, \omega$. Hence

$$\begin{array}{ll} \boldsymbol{k} \cdot \boldsymbol{E} = 0 & \boldsymbol{k} \cdot \boldsymbol{B} = 0 \\ \boldsymbol{k} \times \boldsymbol{E} = \omega \, \boldsymbol{B} & \boldsymbol{k} \times \boldsymbol{B} = -\mu_0 \, \epsilon_0 \, \omega \, \boldsymbol{E} \end{array}$$

These equations tell us that $\boldsymbol{k}$, $\boldsymbol{E}$, and $\boldsymbol{B}$ are mutually orthogonal. Also, the magnitude tells us that $E = c \, B$.

(c) An electromagnetic plane wave propagating in the $+x$ direction is polarized in the y direction. Write down an expression for its electric field. [1]

$$\boldsymbol{E}(\boldsymbol{r}, t) = (0, E_0, 0)\, e^{i(k_x x - \omega t)}$$

(d) Write down the components of the magnetic field amplitude ($\boldsymbol{B}_0$) for this electromagnetic plane wave. [1]

Using $\boldsymbol{k}\times\boldsymbol{E} = \omega\boldsymbol{B}$ and $\frac{\omega}{k} = c$ we obtain

$$\boldsymbol{B}(\boldsymbol{r}, t) = \left(0, 0, \frac{E_0}{c}\right) e^{i(k_x x - \omega t)}$$

Friday, October 16, 2009