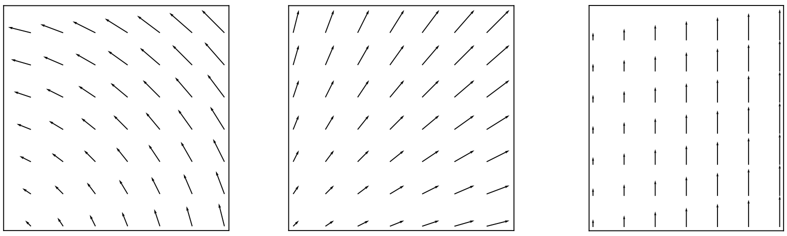
ElectroMagnetism — Assignment 2

Solutions are to be handed in by 5pm, Monday September 14

1. Consider the following three vector fields:



(1) (2) (3)

(a) Which of these is *solenoidal*, *i.e.*, $\nabla \cdot \boldsymbol{v} = 0$? [2]

(b) Which of these is *irrotational*, *i.e.*, $\nabla \times \boldsymbol{v} = \boldsymbol{0}$? [2]

2. A hollow spherical shell of inner radius a and outer radius b has charge density $\rho = k / r^2$ in the region $a \le r \le b$. Compute and display the (magnitude of the) electric field and the potential for $r \ge 0$. Check that $\nabla \cdot \boldsymbol{E} = \rho / \epsilon_0$. [4]

3. Consider a thin evacuated spherical shell of radius R which has surface potential

$$V(R, \theta) = k \sin^2\left(\frac{\theta}{2}\right) \tag{1}$$

where k is a constant and $0 \le \theta \le \pi$ is the angle in spherical polar coordinates measured from the z-axis. In the case of azimuthal symmetry, the *general solution* to Laplace's equation $\nabla^2 V = 0$ is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \tag{2}$$

where $P_l(x)$ are the Legendre polynomials: $P_0(x) = 1, P_1(x) = x, \ldots$

Use equations (1) and (2) and the *orthogonality* of the Legendre polynomials to determine the *internal* potential, *i.e.*, $V(r, \theta)$ for $r \le R$. Hint: $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\,(1 - \cos(\theta)) = \frac{1}{2}\,(P_0(\cos(\theta)) - P_1(\cos(\theta)))$. [6]

4. For a point charge:

(a) Calculate $\nabla \times \boldsymbol{E}$ directly. [1]

(b) Compute $\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$, in spherical coordinates using $d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin(\theta)\,d\phi\,\hat{\boldsymbol{\phi}}$. [1]

(c) Show that the integral in (b) around *any* closed path is zero. [1]

(d) Compute $\nabla \times \boldsymbol{E}$ from (c) by applying Stokes' theorem. [1]

(e) Show how you can conclude that $\nabla \times \boldsymbol{E} = \boldsymbol{0}$, for *any* static charge distribution. [2]

5. Two vector identities.

(a) Show that $\nabla \times (\nabla \psi) = \mathbf{0}$, using Stokes' theorem.[2]

(b) Show that $\nabla \cdot \nabla \times \boldsymbol{A} = 0$, for $\boldsymbol{A}$ an arbitrary differentiable vector field by applying Stokes' theorem to an arbitrary *closed* surface S and then using Gauss' theorem. [3]

6. Maxwell's equations in free space ($\boldsymbol{J} = \mathbf{0}$ and $\rho = 0$) read

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t} \quad \nabla \times \boldsymbol{B} = \mu_0 \, \epsilon_0 \, \frac{\partial E}{\partial t}$$

(a) Using Maxwell's equations derive the wave equation for $\boldsymbol{B}$. [3]

(b) Show that the divergence of the second pair of Maxwell's equations are consistent by using the vector field identity $\nabla \cdot \nabla \times \boldsymbol{A} = 0$ which holds for *any* vector field $\boldsymbol{A}$. [2]

7. Consider the plane-wave electric field $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$, where $\boldsymbol{k} = (k_x, k_y, k_z)$ is the wave-vector, $\boldsymbol{r} = (x, y, z)$, and ω is the angular frequency.

(a) Show that $\boldsymbol{E}(\boldsymbol{r}, t)$ is a solution of the wave equation. Obtain a relation between $k = |\boldsymbol{k}|$, ω, and the phase velocity. [2]

(b) Use $\boldsymbol{E}(\boldsymbol{r}, t)$, (plus an analogous expression for $\boldsymbol{B}$), to convert Maxwell's equations from vector differential equations to plain vector equations. What is the physical meaning of these vector relations? [3]

(c) An electromagnetic plane wave propagating in the $+x$ direction is polarized in the y direction. Write down an expression for its electric field. [1]

(d) Write down the components of the magnetic field amplitude ($\boldsymbol{B}_0$) for this electromagnetic plane wave. [1]

Monday, August 31, 2009